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Higher-order and non-local effects in homogenisation of periodic media

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Higher-order and Non-local Effects in Homogenisation of Periodic Media

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Abstract

The work aims at rigorous derivation of the so-called "size effects" widely documented experimentally in the overall behaviour of "microstructured" solid media when the "scale interaction" effect becomes a factor.

With this aim we give a mathematically rigorous derivation of higher-gradient and non-local effects in the behaviour of heterogeneous periodic continuum media. The mathematical theory of homogenisation is utilized for this, the core technique being the method of multiple-scale asymptotic expansions. We use it in a combination with variational approach to construct via the so-called "variational truncation" the higher-order homogenised equations for a linear uniformly elliptic periodic medium without boundary. The higher-order homogenised equations are proved to have unique solutions, which provide better approximations to the actual solution in terms of the improved rate of convergence of energies and the solutions themselves. An implementation of the results of the analysis is given for derivation of higher-order constitutive relations, which are in agreement with the phenomenological strain-gradient theories.

The construction is then carried over to the case of a (scalar) nonlinear with growth p, $1 < p < \infty$, "quadratically" uniformly elliptic periodic medium under certain standard technical conditions. Full asymptotic expansion is constructed and rigorously justified via the error analysis. It is further shown that the condition of uniform ellipticity may sometimes be relaxed. Applications to the study of power-law constitutive relations are presented: in the dimension two "univalence" results based on the index theory ensure that the construction goes through provided the gradient of the classical homogenised solution does not vanish.

On the other hand, it is further shown that *non-uniformly elliptic* media may exhibit spatial non-locality in the homogenised limit. This is established for a 3D fibre-reinforced conducting medium with highly anisotropic fibres of "double-porosity" type. The limit equations exhibit non-locality along the fibres, which is rigorously established using two methods: two-scale asymptotic expansions and two-scale convergence. An important ingredient of the analysis is a "high-contrast" version of Poincaré's inequality, which is also derived in this work.

In conclusion, an interpretation of the higher-gradient and non-local effects from a unifying standpoint is given. This is done by first introducing the stress-strain relation between the "ensemble mean" stress and strain for a simple random medium of translations of the periodic medium. It is shown that the operator is non-local and that it "localizes" when the cell size ε becomes small, having the higher-gradient expansion as its rigorous asymptotics (conforming formally with the gradient-type approximation of the non-local operator). On the other hand, if the contrast between the phases increases simultaneously with diminishing ε, for the critical (double-porosity type) scaling the limit stress-strain relation remains non-local as emerges in the double porosity models.

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Contents

Introduction

0.1 Motivation of the work: size effects

The main motivation for the work presented in the thesis comes from the desire to account for various so-called *size effects* observed in the behaviour of elastic and plastic materials with microstructure. In this introductory section we present a heuristic argument giving an idea of the issue to be addressed.

To outline the starting point, consider an experiment where one of two identical thin wires with diameter of order 100 μm undergoes a torsional deformation while the other is subjected to a uniaxial tensile deformation, as reported in Fleck *et al* [24] (see also Fleck & Hutchinson [23]). Then other pairs of identical wires are taken, whose diameters decrease down to a few microns. In the torsion experiment the torsional response is measured as a function of twist per unit length κ (Figure 1a), and in the tension experiment the stress is measured versus logarithmic strain (Figure 1b). The experiment has demonstrated the following:

1) In the first case (torsion) the normalized torsional rigidity Q/a^3 shows considerable dependence on the radius of the wire, whereas in the second case (tension) the effect of changing the radius on the stress-strain relation is negligible;

2) In the case of torsion the influence of the radius of the wire on its (normalized) strength increases as the radius diminishes.

In other words, in the case of torsion a size effect, namely the dependence of the (appropriately normalized) response on the representative scale L of the deformation field (in this particular situation it is the radius of the wire a), is observed unlike in the case of uniform stretching. The torsion size effect gets stronger as the scale L gets smaller.

The above wire torsion experiment is one of many examples of the experimentally documented size effects. A typical cartoon is displayed on Figure 2. (For the above example $M = Q/a^3$, $L/l = a/l$, for a fixed κ.) When the deformation scale L is much larger than some "intrinsic" length scale of the material l, the response is close to the one without the size effect (upon appropriate rescaling, the horizontal dashed line in Figure 2). However, when L/l is still large but "not too large", a departure from M_0 is observed as symbolized by the solid curve in Figure 2. Introducing a small parameter $\varepsilon = (L/l)^{-1}$, one can speculate that M_0 corresponds to a "homogenised limit" $\varepsilon \to 0$

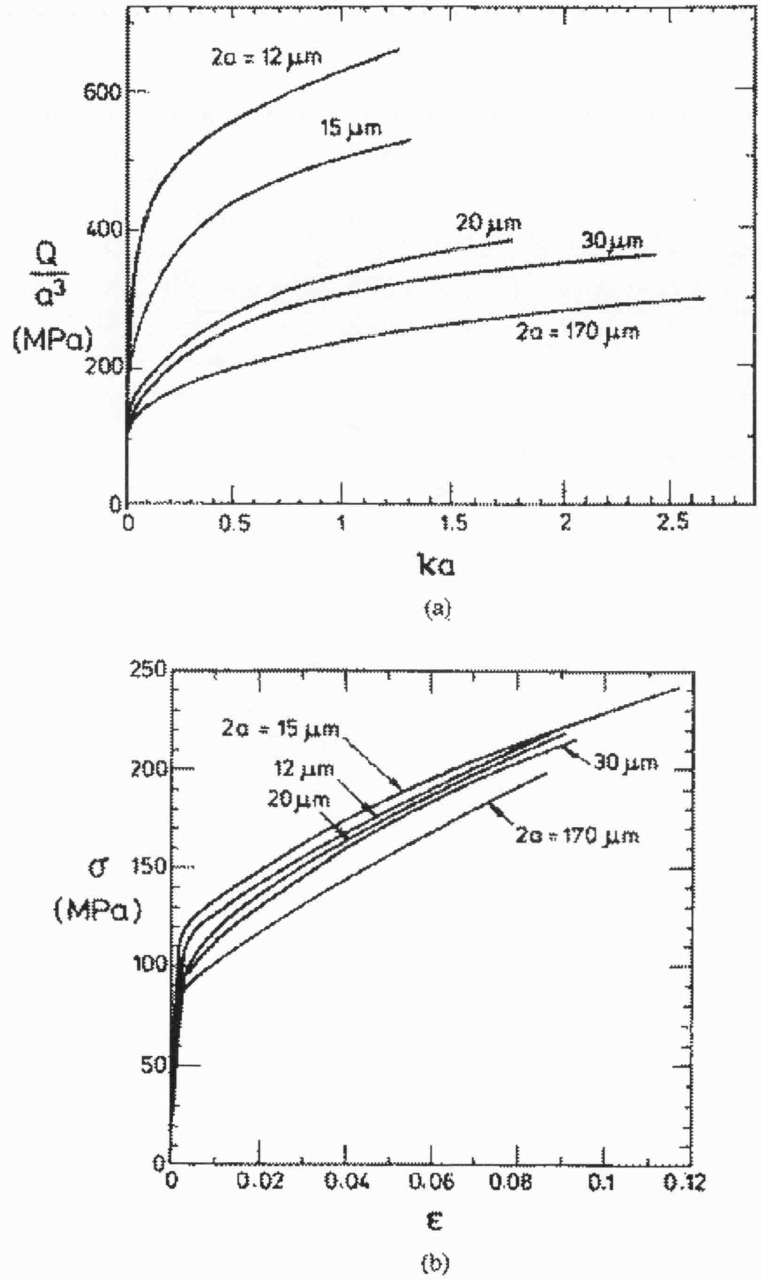

Figure 1: a) Torsional response of copper wires of diameter $2a$ in the range $12\mu m$ to $170\mu m$. Both the torque Q and the twist per unit length κ are scaled by the wire radius a. If the constitutive law were independent of strain gradients, the plots of normalised torque Q/a^3 versus κa would all lie on the same curve. b) True stress σ versus logarithmic strain ε tension data for copper wires of diameter $2a$ in the range $12\mu m$ to $170\mu m$. There is a negligible effect of wire diameter on the behaviour. *The figure and the caption are courtesy of Fleck and Hutchinson [23].*

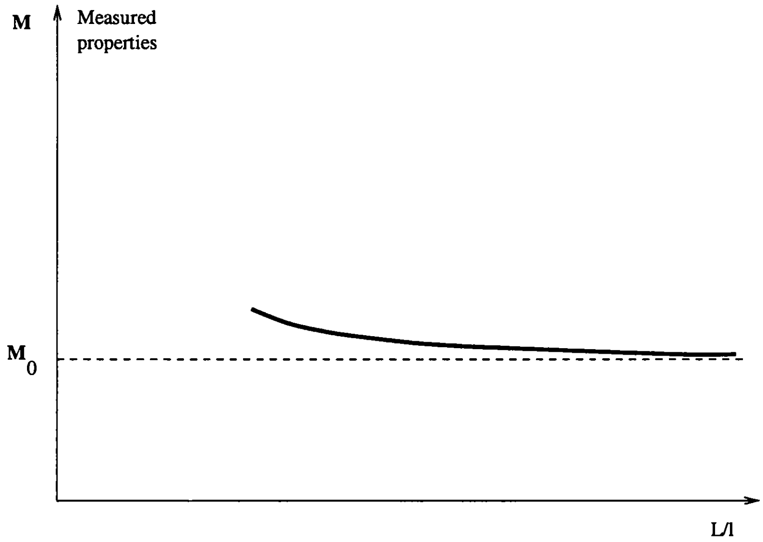

Figure 2: The measured properties $\mathbf{M}$ of a heterogeneous material can be approximated by the measured properties $\mathbf{M}_0$ of a certain homogeneous material when the ratio L/l is large (mathematically, $\mathbf{M} = \mathbf{M}(L/l) \to \mathbf{M}_0$ as $L/l \to \infty$). As the scales L and l become comparable, like in the case of thin wires, there is a departure from the value $\mathbf{M}_0$.

of the separation between the two physical scales. Then, the above departure from $\mathbf{M}_0$ must have something to do with the *asymptotic corrector* to $\mathbf{M}_0$ with respect to the small ε.

Fleck and Hutchinson [23] as well as many of their predecessors have argued that this kind of size effect is associated with the presence of a gradient of strain in the wire along the radius, and that therefore the gradient of the strain should be introduced into the constitutive relation. Indeed, if the wire were a uniform continuum with stress-strain relation of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{e})$, where $\boldsymbol{\sigma}$ is the stress and $\boldsymbol{e}$ is the strain (either geometrically linear or even geometrically nonlinear), *i.e.* without any dependence on higher gradients of the strain, then in the case of the torsional deformation we would not observe any size effects (*i.e.* all the curves in Figure 1a would coincide), as simple dimensional analysis suggests. Indeed, in this case two "identical" deformations of wires of radii a and $a' = \lambda a$, $\lambda > 1$ (the second deformation is an exact copy of the first one "blown up" by λ times) would give us

$$\boldsymbol{x}' = \lambda \boldsymbol{x}, \quad u'(\boldsymbol{x}') = \lambda u(\boldsymbol{x}), \tag{1}$$

where $u(\boldsymbol{x})$ and $u'(\boldsymbol{x}')$ are the displacements of the points $\boldsymbol{x}$ and $\boldsymbol{x}'$ of the respective wires. From the relations (1) we immediately get $\boldsymbol{e}'(\boldsymbol{x}') = \boldsymbol{e}(\boldsymbol{x})$ and therefore $\boldsymbol{\sigma}'(\boldsymbol{x}') = \boldsymbol{\sigma}(\boldsymbol{x})$. As a result, necessarily, $Q'/(a')^3 = Q/a^3$ with no size effect.

To repair this, numerous investigators assume dependency on higher gradients of strain in the constitutive relation. Then the size effects described above become justifiable, as the following relation obviously holds

$$\nabla_{x'} \boldsymbol{e}'(\boldsymbol{x}') = \lambda^{-1} \nabla_x \boldsymbol{e}(\boldsymbol{x}),$$

and thus the constitutive relation becomes capable of capturing the said size effects.[1] Fleck and Hutchinson [23] and others have been fitting the strain gradient term "phenomenologically" to match the experiment in the best possible way. Their approach is based on the idea that the above lengthscale l must have a clear physical interpretation: it is associated with the "scale" of the underlying microstructure, *e.g.* dislocations spacing in plasticity.

The issue we hope to address in this work is how constitutive relations involving higher gradients of strain can be derived mathematically rigorously from the underlying microstructure of a material. Realistic microstructure may often be *discrete* (*e.g.* dislocations in plasticity), which introduces a substantial complication for purposes of their homogenisation. Instead, in this work we consider a heterogeneous continuum material that for simplicity has two length scales, the representative scale L of the deformation field, which is macroscopic, and the microscopic scale l of heterogeneities of the material. We hope that the two-scale continuum model reflects the essence of scale interaction behind the size effects. Then, it is well known from the theory of homogenisation that when the ratio L/l is very large, one can often claim that the original heterogeneous medium behaves "almost like" a homogeneous medium. The effective parameters of the limiting material can sometimes be estimated or even calculated precisely, and the degree of "closeness" of its properties to the properties of the original material can be controlled. On the other hand, when L/l is not too large, there is a departure in the properties of the original heterogeneous medium from the properties of the homogenised limit, with the difference in properties getting stronger as L/l gets smaller, as shown in Figure 2. The chief motivation of this work is therefore to derive constitutive models capable of predicting the above size effects by going to "higher orders" in homogenisation of the microscale. A major technical tool used for this is the mathematical theory of homogenisation, which we review next.

0.2 A brief historical survey of homogenisation

In this section we present a short historical account of the mathematical theory of homogenisation. We do not aim at mentioning all relevant references, and by no means our task here is to produce a comprehensive review of the subject, which is extremely vast. Rather, the literature mentioned here refers mostly to the papers that have been the source of our own understanding of homogenisation and that influenced (directly or indirectly) the results presented in this thesis.

[1] Note, that the equilibrium equation in this case involves derivatives of the displacement $u(\boldsymbol{x})$ of orders higher than second.

The mathematical theory of homogenisation took its origin probably in the work by De Giorgi and Spagnolo [20], where the authors proved the first theorem on passing to the limit in linear partial differential equations with uniformly elliptic rapidly oscillating periodic coefficients (as well as parabolic equations) when the small size of the unit cell tends to zero. Their proof was fairly complicated and was based on results of Spagnolo [46], [47], who introduced the notion of G-convergence associated with sequences of symmetric linear partial differential equations and studied its properties.

Shortly afterwards Tartar [49] suggested a direct proof of the homogenisation theorem using the so-called energy method or compensated compactness theory. His approach was based on using in the course of passing to the limit a special choice of test functions in the variational formulation of the equation, in combination with the now-classical compensated compactness lemma.

About the same time Sanchez-Palencia [37] and Bakhvalov [7], [8] used the method of multiple-scale asymptotic expansions for construction of the homogenised equation. Sanchez-Palencia only did formal asymptotics but Bakhvalov has also proved that the solution of the homogenised equation is the limit of the solutions to the original heterogeneous problems as the small parameter tends to zero, and provided the corresponding error estimates. The concept of the method of asymptotic expansions (powerful from both heuristic and rigorous points of view) is dated back to Krylov & Bogoliubov [29] and Bogoliubov & Mitropolsky [14] and has been successfully used in numerous problems involving multiple scales since then. The related now-classical idea is to seek an asymptotic expansion of the solution to a given problem with a small parameter in powers of the small parameter with coefficients that "separate" the fast and the slow scales of the problem.

Results similar to [37], [7], [8] were obtained a little later but independently by Keller [27] and Bensoussan *et al.* [13]. The applications and extensions of this general method in the theory of homogenisation were further developed and summarised in Sanchez-Palencia [38] and Bakhvalov & Panasenko [11].

Up to 1975, homogenisation results had been given only for linear problems. However, very soon De Giorgi and Franzoni [19] developed the theory of Γ-convergence of nonlinear functionals, which provided a rigorous way to perform homogenisation in those nonlinear problems that admit variational formulation with a convex (not necessarily quadratic) functional. Also about the same time, Bakhvalov [9] applied his original idea of multiple-scale asymptotic expansions to a general nonlinear case. However, he constructed only formal asymptotic expansions, thus without establishing any convergence results.

The next major landmark in the theory of homogenisation is the work by Murat and Tartar [34] who introduced the notion H-convergence of monotone operators and established its main properties. This method in particular allows to perform homogenisation in linear elliptic equations whose coefficients are not necessarily symmetric, thus containing as a particular case the results of De Giorgi and Spagnolo mentioned above.

On the other hand, this method allows to treat a range of nonlinear problems that do not necessarily admit variational formulation and therefore are not covered by the theory of Γ-convergence.

Around one decade later, another method applicable to a large set of problems whose solutions display a "two-scale" behaviour was developed, namely the method of two-scale convergence. This technique was introduced in the work of Nguetseng [35] and then developed further and illustrated by numerous applications in the work by Allaire [4]. This method enhanced understanding of oscillations in the process of homogenisation by introducing a suitable notion of convergence for functions depending on a "microscopic" variable in addition to the slow "homogenised" variable. In this sense, the two-scale convergence method is somewhat close to the two-scale asymptotic approach. A novel technical component of this method was a two-scale compactness lemma.

The two-scale convergence technique is powerful enough to perform homogenisation in a broad range of rapidly oscillating partial differential equations. In particular, it recovered previously established results on averaging of variational problems (including non-quadratic ones), of monotone operators and of problems in perforated domains. Importantly, it allowed to treat problems not satisfying the uniform ellipticity condition, *e.g.* high-contrast media exhibiting double-porosity effects. Moreover, as was shown in a recent work by Zhikov [60], it can be used for treating an even wider variety of problems than was thought before, by generalizing it onto the case of two-scale convergence in periodic Lebesgue spaces associated with arbitrary measures, with applications *e.g.* to homogenisation of networks and junctions. In particular, the two-scale method finds new exciting applications in the theory of elastic thin structures, see Zhikov [61].

0.3 An overview of the work presented in the thesis

The subject area of the thesis can be defined as non-standard (strain gradient or non-local) effects in the overall behaviour of multiple-scale media. This topic is motivated by numerous physical and engineering applications involving scale interaction effects (see Section 0.1). At the same time the complexity of the problems involved requires application and development of advanced mathematical methods, *e.g.* mathematical theory of homogenisation, asymptotic methods, variational methods, multiple-scale convergence, *etc.* Below, we overview briefly the work completed in this area in the course of the PhD project (see also Smyshlyaev & Cherednichenko [42], Cherednichenko & Smyshlyaev [16], Cherednichenko *et al* [17]) and outline the structure of the thesis.

In Chapter 1, we start with the problem of rigorous derivation of higher order terms in the asymptotic behaviour of periodic composites with the periodicity cell of small size ε (see Smyshlyaev & Cherednichenko [42]). The problem is considered in the context of anti-plane shear of a linear elastic composite.

In Section 1.1 the formulation of the problem is given and the classical two-scale asymptotic expansion of its solution is reviewed following Bakhvalov and Panasenko [11]. We assume that the body force $f(\boldsymbol{x})$ applied to the composite is $\mathbf{T}$-periodic, where $\mathbf{T} = [-T, T]^2$, $T > 0$; $\varepsilon > 0$ is such that $\varepsilon^{-1}T$ is a positive integer. The problem is to study the properties of the $\mathbf{T}$-periodic solution to the following classical elliptic equation (comma in subscript means differentiation)

$$-\left(A_{ij}(\boldsymbol{x}/\varepsilon)u^\varepsilon_{,j}\right)_{,i} = f(\boldsymbol{x}), \quad \int_{\mathbf{T}} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \quad \int_{\mathbf{T}} u^\varepsilon(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0. \tag{2}$$

Here $(A_{ij}(\boldsymbol{y}))^2_{i,j=1}$ is a symmetric Q-periodic uniformly elliptic matrix of elements of the elastic tensor of the medium, $Q = [0,1]^2$. In the classical theory of periodic homogenisation, there is a well-known procedure of identifying with the rapidly oscillating problem (2) a homogenised problem

$$-(h_{ij}v_{,j})_{,i} = f, \quad \int_{\mathbf{T}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0,$$

where the constant homogenised coefficients h_{ij} are explicitly found from an appropriate canonical unit cell problem. The concept of homogenised solution is rigorously supported by convergence of solutions ($u^\varepsilon \overset{W^{1,2}}{\rightharpoonup} v$) and by convergence of energies $\left(E^\varepsilon(u^\varepsilon) \to E^0(v)\right)$.

In Section 1.2 we develop a combination of asymptotic (see *e.g.* Bakhvalov & Panasenko [11]) and variational methods and, as a result, derive higher-order homogenised equations. Namely, for $K \geq 2$ we construct a *homogenised equation of order* $2K$ of the form

$$-h_{ij}v_{,ij} - \sum_{l=3}^{2K} \varepsilon^{l-2} \sum_{|k|=l} h_k^{(K)} D^k v = f.^2 \tag{3}$$

Here $h_k^{(K)}$ are "higher-order" homogenised coefficients.

We prove that the higher-order problem (3) is well-posed and that its unique solution $v_K(\boldsymbol{x})$ provides a better approximation to $u^\varepsilon(\boldsymbol{x})$, both in terms of the energy convergence $\left(|E^\varepsilon(u^\varepsilon) - E^{(K)}(v_K)| \leq C(K)\varepsilon^{2(K-1)}\right)$ and convergence of solutions $\left(\|\bar{u}^\varepsilon - v_K\|_{L^2(\mathbf{T})} \leq C(K)\varepsilon^{K-1}\right.$, where $\bar{u}^\varepsilon$ is the average for a "translated" family of the problems of type (2), see Smyshlyaev & Cherednichenko [42]). This result is obtained by employing the full asymptotic expansion for the solution of (2) in the "double-series"

[2] Here $k := k_1...k_l$ is a multi-index, where $k_j = 1, 2$; $j = 1, ..., l$; $|k| = l$ is the "length" of k and $D^k v = v_{,k_1...k_l}$. The summation with respect to $|k| = l$ in (4) means the sum of 2^l terms for all possible k of the length l.

form (see Bakhvalov & Panasenko [11]):

$$u^{\varepsilon}(\boldsymbol{x}) \sim v(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{|k|=l} N_k(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x},\varepsilon),^3 \tag{4}$$

where

$$v(\boldsymbol{x},\varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}), \tag{5}$$

and then using the truncated form of (4) for the restricted set of trial fields in the translation-averaged variational formulation. This results in a (convex) variational problem for v, for which (3) is the Euler-Lagrange equation. These results, while being mathematically rigorous, allow at the same time an efficient numerical implementation by solving appropriate higher-order unit cell problems, *e.g.* via finite elements, which has been executed recently (see Peerlings & Fleck [36]).

In Section 1.3, the implementation to derivation of higher-order effective constitutive relations in linear elasticity is given. The possible asymptotic and variational approaches are discussed and compared.

In Chapter 2 we consider the problem of derivation of higher-order terms in the asymptotic behaviour of the *quasilinear* rapidly oscillating problem, namely

$$-\mathrm{div}\, \boldsymbol{j}\Big(\boldsymbol{x}/\varepsilon, \nabla u^{\varepsilon}(\boldsymbol{x})\Big) = f(\boldsymbol{x}), \ \int_{\mathbf{T}} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \ \int_{\mathbf{T}} u^{\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \tag{6}$$

where $\boldsymbol{x} \in \mathbf{T} = [-T,T]^d$, $T > 0$, function $\boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})$ is Q-periodic in $\boldsymbol{y}$, $Q = [0,1]^d$, and $\varepsilon > 0$ is a small parameter such that $\varepsilon^{-1}T$ is a positive integer.

In Section 2.1 the rigorous mathematical formulation of the problem is given. We consider the case when there exists an infinitely smooth potential $W = W(\boldsymbol{y},\boldsymbol{e})$ such that $\boldsymbol{j}(\boldsymbol{y},\boldsymbol{e}) = \nabla_{\boldsymbol{e}} W(\boldsymbol{y},\boldsymbol{e})$ and impose standard requirements on the function $W(\boldsymbol{y},\boldsymbol{e})$ that are given below.

1) The function $W(\boldsymbol{y},\boldsymbol{e})$ satisfies a growth condition as follows

$$-A_1 + B_1|\boldsymbol{e}|^p \le W(\boldsymbol{y},\boldsymbol{e}) \le A_2 + B_2|\boldsymbol{e}|^p \ \text{ for any } \boldsymbol{y} \in Q, \boldsymbol{e} \in \mathbf{R}^d \tag{7}$$

with some positive constants A_1, A_2, B_1, B_2 and $p > 1$.

2) The function $W(\boldsymbol{y},\boldsymbol{e})$ is uniformly strictly convex, *i.e.* the following inequality holds with some constant $\nu > 0$

$$\frac{\partial^2 W(\boldsymbol{y},\boldsymbol{e})}{\partial e_i \partial e_j} \eta_i \eta_j \ge \nu \eta_i \eta_i$$

[3] The "microstructural" functions $N_k(\boldsymbol{y})$ are Q-periodic in $\boldsymbol{y}$ and have zero mean over Q. They are uniquely constructed from appropriate higher-order canonical unit cell problems. In turn, $h_k^{(K)}$ are explicitly expressed via $N_k(\boldsymbol{y})$.

for any $\boldsymbol{y} \in Q, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\eta} = (\eta_1, ..., \eta_d) \in \mathbf{R}^d$.

3) The function $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{y}, \boldsymbol{e})$ is strongly monotonic with respect to $\boldsymbol{e}$:

$$\Big(\nabla_e W(\boldsymbol{y}, \boldsymbol{e}_1) - \nabla_e W(\boldsymbol{y}, \boldsymbol{e}_2)\Big) \cdot \Big(\boldsymbol{e}_1 - \boldsymbol{e}_2\Big) \geq \alpha |\boldsymbol{e}_1 - \boldsymbol{e}_2|^p, \quad \alpha > 0,$$

where p is the same as in (7), for every $\boldsymbol{y} \in Q$ and all $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbf{R}^d$.

Under these conditions, in Section 2.2 we derive a "double-series" asymptotic expansion of the solution $u^\varepsilon(\boldsymbol{x})$ to the problem (6) as follows (see also Cherednichenko & Smyshlyaev [16])

$$u^\varepsilon(\boldsymbol{x}) \sim v(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l u_l\Big(\boldsymbol{x}/\varepsilon, \nabla v(\boldsymbol{x}, \varepsilon), \nabla\nabla v(\boldsymbol{x}, \varepsilon), ..., \nabla^l v(\boldsymbol{x}, \varepsilon)\Big), \tag{8}$$

where

$$v(\boldsymbol{x}, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}). \tag{9}$$

Functions $u_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big)$ are Q-periodic in $\boldsymbol{y}$ and have zero mean over Q; functions $v_s(\boldsymbol{x})$ are $\mathbf{T}$-periodic with zero mean over $\mathbf{T}$, and do not depend on the fast variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$.

Further, in Section 2.3 the expansion (8)–(9) is rigorously justified, *i.e.* the estimate $\|u^\varepsilon(\boldsymbol{x}) - u^{(K)}(\boldsymbol{x}, \varepsilon)\|_{W^{1,p}_{0,per}(\mathbf{T})} \leq C(K)\varepsilon^{K-1}$ is proved, where $u^{(K)}$ is the truncation of (8)–(9).

Section 2.4 is devoted to the discussion of further aspects of the problem. Namely, the issues of infinite-order homogenised solution and higher-order homogenised variational problems are addressed, and applications to non-uniformly elliptic problems, in particular the power-law case, are presented.

It is important to emphasize that the above results have been obtained under the assumption of uniform ellipticity (both in the linear and in the nonlinear cases). In this case the higher-order terms are merely correctors to the classical homogenised limits. One can notice that the correctors' "strength" increases when we are approaching a point of loss of uniform ellipticity. On the other hand it is known (see Allaire & Murat [5], Nguetseng [35], Zhikov [60]) that absence of uniform ellipticity may result in "non-standard" homogenised limits. In the linear case, increasing the *contrast* between phases brings us closer to the loss of uniform ellipticity. With the aim of exploring this, in Chapter 3 we engage next in the study of *non-local* features arising in the structure of the homogenised equations for homogenisation problems with highly contrasting parameters. In particular, we demonstrate a problem of homogenisation resulting in non-local homogenised equations.

In Section 3.1 the formulation of the problem under consideration and its homogenisation are given. We study an elliptic equation with vanishing ellipticity constant when

$\varepsilon \to 0$ as follows

$$-\Big(A^{\varepsilon}_{ij}(\boldsymbol{x}/\varepsilon)u_{,j}\Big)_{,i} + \lambda u = f(\boldsymbol{x}), \quad \lambda > 0, \tag{10}$$

where $f(\boldsymbol{x})$ is $\mathbf{T}$-periodic, $\mathbf{T} = [-T, T]^3$, $T > 0$, and has zero mean over $\mathbf{T}$. In equation (10), the matrix of coefficients $\Big(A^{\varepsilon}_{ij}(\boldsymbol{y})\Big)$ is defined by the following formula

$$\Big(A^{\varepsilon}_{ij}(\boldsymbol{y})\Big) = \begin{cases} \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_1, \end{cases}$$

where $F_0 = \tilde{F}_0 \times [0,1]$ and $F_1 = \tilde{F}_1 \times [0,1]$, $\tilde{F}_0$ and $\tilde{F}_1$ being $[0,1]^2$-periodic sets with Lipschitz boundaries; $\tilde{F}_0 \cap \tilde{F}_1 = \emptyset$ and $\tilde{F}_0 \cup \tilde{F}_1 = \mathbf{R}^2$. We assume that the set F_1 is connected (and therefore the associated measure $d\boldsymbol{x}|_{F_1}$ is "ergodic", see Zhikov [60]) and that its volume fraction $|F_1 \cap Q|$ does not vanish, which allows us to use techniques originated by Zhikov [60].

To find the homogenised behaviour of the solution to the equation (10) we use the method of two-scale convergence originated in Nguetseng [35] and further developed in Allaire [4] and Zhikov [60]. We use standard two-scale compactness arguments to show that

$$u^{\varepsilon}(\boldsymbol{x}) \xrightarrow{2} u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x}, y_1, y_2),$$

where $\xrightarrow{2}$ denotes two-scale convergence. The $\mathbf{T}$-periodic functions $u^{(1)}$ and w satisfy the following system of elliptic equations[4]

$$\begin{cases} -\mathrm{div}\Big(A^{hom}\nabla u^{(1)}\Big) - \dfrac{\partial^2 \langle w\rangle}{\partial x_3^2} + \lambda\Big(u^{(1)} + \langle w\rangle\Big) = f, & \boldsymbol{x} \in \mathbf{T}, \\ -\dfrac{\partial^2 w}{\partial y_1^2} - \dfrac{\partial^2 w}{\partial y_2^2} - \dfrac{\partial^2 w}{\partial x_3^2} - \dfrac{\partial^2 u^{(1)}}{\partial x_3^2} + \lambda\Big(u^{(1)} + w\Big) = f, & \boldsymbol{y} \in F_0 \cap Q, \end{cases} \tag{11}$$

together with the condition

$$w(\boldsymbol{x}, y_1, y_2)\Big|_{y \in F_0 \cap Q} = 0. \tag{12}$$

Here, the homogenised matrix A^{hom} is given by the following formula

$$A^{hom} = \begin{pmatrix} \int\limits_{F_1\cap Q} \Big(1 + (N_1)_{,1}(y_1,y_2)\Big)dy_1dy_2 & \int\limits_{F_1\cap Q} (N_1)_{,2}(y_1,y_2)dy_1dy_2 & 0 \\ \int\limits_{F_1\cap Q} (N_2)_{,1}(y_1,y_2)dy_1dy_2 & \int\limits_{F_1\cap Q} \Big(1 + (N_2)_{,2}(y_1,y_2)\Big)dy_1dy_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[4] $\langle\cdot\rangle$ denotes averaging over $Q = [0,1]^3$.

where $N_1(y_1, y_2)$ and $N_2(y_1, y_2)$ are the solutions of the corresponding two-dimensional unit cell problems.

As an important analytic point, in Section 3.2 we establish (developing certain ideas of Allaire and Murat [5]) the following Poincaré-type inequality for high contrast media

$$\|u\|_{L^2(\mathbf{T})} \leq C\Big(\|\nabla u\|_{L^2(\mathbf{T}\cap F_1^\varepsilon)} + \varepsilon\|\nabla u\|_{L^2(\mathbf{T}\cap F_0^\varepsilon)}\Big), \quad u \in H^1(\mathbf{T}), \quad \int_{\mathbf{T}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \quad (13)$$

where the constant C does not depend on ε. In this inequality $\mathbf{T} = [-T, T]^d$, $T > 0$; F_0 is a Q-periodic set, $Q = [0,1]^d$; $F_1 = \mathbf{R}^d \setminus F_0$; $F_0^\varepsilon = \varepsilon F_0$, $F_1^\varepsilon = \varepsilon F_1$, where $\varepsilon > 0$ is such that $\varepsilon^{-1}T$ is a positive integer. The inequality (13) allows us to treat the case when $\lambda = 0$ in (10) and to perform rigorous asymptotic analysis of (10).

In Section 3.3 we show that the weak solution $(u^{(1)}, w)$ of the problem (11)–(12) is infinitely smooth. Eliminating w we find that the function $u^{(1)}$ satisfies the following integro-differential equation (here λ is set to zero):

$$-\mathrm{div}\Big(A^{hom}\nabla u^{(1)}\Big) - \mathfrak{f}_0^2\langle G\rangle_{\tilde{y}',\tilde{y}} \overset{x_3}{*} \frac{\partial^4 u^{(1)}}{\partial x_3^4} = f + \mathfrak{f}_0^2\langle G\rangle_{\tilde{y}',\tilde{y}} \overset{x_3}{*} \frac{\partial^2 f}{\partial x_3^2}, \qquad (14)$$

where $G = G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3)$, $\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}' \in \tilde{F}_0, x_3 \in \mathbf{R}$ is the Green's function for the "rescaled" fibres, and $\mathfrak{f}_0 := |F_0 \cap Q|$ is the volume fraction of the fibres. Thus, (14) exhibits the non-local homogenised limit for the problem (10). Also in Section 3.3 we explain how the homogenised system (11) derived in Section 3.1 can be obtained using the method of two-scale asymptotic expansion. The asymptotic series is further rigorously substantiated by corresponding remainder estimates. The presentation of the non-local homogenised stress-strain relation and the derivation of an explicit formula for the kernel of the corresponding convolution operator in the case when $\tilde{F}_0$ is a disc, conclude Section 3.3.

It is instructive that this rigorous result (established using both the technique of two-scale convergence and the method of asymptotic expansion) can also be re-derived formally, from the strain gradient asymptotics of the solution to the problem

$$-\Big(A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)u_{,j}\Big)_{,i} = f(\boldsymbol{x}),$$

where the matrix of coefficients is initially determined by the following formula

$$\Big(A_{ij}^{\varkappa}(\boldsymbol{y})\Big) = \begin{cases} \begin{pmatrix} \varkappa & 0 & 0 \\ 0 & \varkappa & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_1. \end{cases}$$

The discussion of this is the subject of Chapter 4. Fixing first $\varkappa$ and treating ε as a parameter we arrive at the asymptotic expansion (4)–(5), where v_s and N_k depend on $\varkappa$. Further, letting $\varkappa$ be small we observe that when $\varkappa$ is of order ε^2 the asymptotic series (4)–(5) "breaks up": all the terms become of equal "strength". It turns out that the main order terms constitute a certain expansion to the two-scale limit $u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x}, x_1/\varepsilon, x_2/\varepsilon)$.

It seems to be a key feature of double-porosity models that they retain the general property of non-locality for an arbitrary periodic medium derived in Section 4.1 "in the very limit" when $\varepsilon \to 0$, as shown in Section 4.3. This fact can be substantiated by considering the "gradient approximation" of the related general convolution operator (Section 4.2), formally substituting $\varkappa = \varepsilon^2$ in it and passing to the limit when $\varepsilon \to 0$, as explained in Section 4.4.

Each chapter (except Chapter 4 that is a kind of summary) commences with an introduction containing a brief survey of the literature related to the subject of the chapter and concludes with a discussion of the results of the chapter.

Part of the technical machinery is given in the main text, the rest being presented in appendices following each chapter. Proofs of propositions, lemmas and theorems end with the symbol $\square$.

Note that unless otherwise specifically indicated, in Chapter 1 and in Chapter 4 we denote $Q = [0,1]^2$, whereas in Chapter 2 we denote $Q = [0,1]^d$, d being a positive integer, and in Chapter 3 we use the notation $Q = [0,1]^3$.

Chapter 1

Rigorous derivation of strain gradient effects in the overall behaviour of periodic heterogeneous media

Introduction

Studying the so called higher-order, non-local or strain gradient effects in the overall behaviour of heterogeneous media has recently become an area of active interest. This has been motivated by the need to account for various scale effects observed in the behaviour of multiple-scale heterogeneous media where the scales are separated widely but not "too widely". Such effects are observed both in the elastic heterogeneous media and in plasticity (see, *e.g.,* a recent review of Fleck and Hutchinson [23] for examples of such scale effects in plasticity, where the smaller scale of dislocation spacing becomes comparable with the larger scale of inhomogeneity of the deformation).

It has been long ago noticed by numerous investigators that an introduction of *length scales* into the continuum constitutive relations is capable of accounting for the above scale effects, whereby the outer scale of inhomogeneity of the deformation competes with the intrinsic length scale entering the constitutive relations. There have been a number of *phenomenological* approaches developed for introducing such length scales (see, e.g., Cosserat & Cosserat [18], Toupin [52], Mindlin [33], Aifantis [1], Fleck & Hutchinson [23], and further references therein). Adapting such a phenomenological constitutive relation, it was possible to account for various size effects, e.g., in torsion of wires (Fleck *et al* [24]) via direct analysis; in the indentation hardness (Shu & Fleck [41]) via finite element simulations; or in the overall behaviour of composites and polycrystals, by deriving rigorous variational upper and lower bounds displaying the scale effects, Smyshlyaev & Fleck [43], [44], [45].

The important question of *how* such length scales in the constitutive behaviour of a heterogeneous medium could be directly derived from the underlying microstructure remains largely unanswered. To address this question, a number of micromechanical approaches have recently been developed to derive the higher-order constitutive relations by averaging in a certain way the physical and geometrical properties of the microstructure. Among others, Willis [57] argued that the actual effective constitutive relations are *non-local* and developed bounds for such relations for a random heterogeneous linear elastic medium via the use of a statistical version of the Hashin-Shtrikman variational principle. Drugan and Willis [21] used this approach in conjunction with a quasi-crystalline type approximation. This reduced the analysis to the use of statistical information of up to two-point correlation functions and allowed the above authors to derive explicit expressions for the associated variational approximation for the non-local overall constitutive relations and, in particular, (in a gradient approximation) for the strain gradient term containing the length scale. An advantage of their approach is that it lies on the variational base, providing approximations which are expected to be best in a certain variational sense. This Hashin-Shtrikman formalism was developed recently further to include the effect of random fluctuations in the applied body force on the non-local effective constitutive relations (Luciano & Willis [31]), and was further specialized to periodic, almost periodic, as well as to more fully random media (Luciano & Willis [32]).

Even though the above variational approaches *are* capable of producing rigorous bounds for non-local effective relations and explicit predictions for the strain gradient terms, the latter still employ *approximations* (of the quasi-crystalline type for the non-local term, subjected to a further gradient approximation) and so the predictions based on them have to be tested by comparison with experiments or with more explicitly modelled microstructures. For the latter, it is natural to start with considering *periodic* microstructures, where the conventional effective moduli are well known to be rigorously characterized in terms of solutions of canonical unit cell problems (see *e.g.* Jikov *et al* [26]), and to attempt to derive the strain gradient terms as rigorous correctors to the effective moduli.

Boutin [15] and Triantafyllidis and Bardenhagen [53] studied higher-order effects in periodic microstructures using the method of asymptotic expansions which has been developed long ago in the mathematical literature on periodic homogenisation (see *e.g.* Bensoussan *et al* [13], Sanchez-Palencia [38], Bakhvalov & Panasenko [11]). The major idea of this method is to construct a two-scale asymptotic expansion of the solution with respect to the small parameter ε which is the ratio of the scale of the microstructure (*e.g.* the size of the periodicity cell) and of the outer length scale (*e.g.* the size of the deformed domain). The asymptotic solution is sought in the form of a function of a rapid (oscillating) variable and of a slow (modulating) variable. After averaging over the rapid variable the homogenised solution varies slowly. This constructs a formal asymptotics of the solution, which may sometimes be made rigorous via the so-called *remainder estimates* showing that the error due to the replacing of the actual solution by the asymptotic one is small for sufficiently small ε. The works of Boutin [15] and Triantafyllidis and Bardenhagen [53][1] construct in this way *formal* asymptotics for the higher-order homogenised stress-strain relations. A well-known difficulty in making this rigorous is the effect of the *boundary*: the boundary layer created by the heterogeneity does not affect the homogenised solution inside the domain in the main approximation but may affect it in higher orders with respect to ε. To avoid dealing with the effect of the boundary, one could consider the infinite periodic medium with the periodic cell of a small size ε in the presence of a *periodic* body force of a *fixed* large period T (*cf.* Bakhvalov & Panasenko [11] who have constructed a full asymptotic expansion in this context and supplemented it by rigorous remainder estimates). Then (if the ratio of T and ε is an integer) the response of the medium is T-periodic and one studies in effect a T-cell periodic solution with a parameter ε tending to zero. This allows to eliminate the effect of the boundary, and, in this way, the formulas of Boutin [15] and Triantafyllidis and Bardenhagen [53] could, in principle, be made rigorous, using the remainder estimates techniques from *e.g.* Bakhvalov and Panasenko. (This still requires some effort and we execute this as a by-product in this chapter.)

Therefore, an essential advantage of the above asymptotic approach is that it is a potentially *rigorous* way of deriving the strain gradient effect, *i.e.* it ensures that the error is small for sufficiently small ε. From the point of view of applications, however, the asymptotic approach has two important limitations. First, it constructs *perturbations* to the conventional homogenised solutions and constitutive relations in sequential powers of a small parameter ε, characterizing the separation of scale. Therefore, it is expected to be good when ε is rather small. If ε is "small but not too small" (which is exactly the case when the scale effects are observed for the larger and smaller scales becoming comparable) the perturbative asymptotic approach may become a less accurate approximation to the actual solution. Second, it does not guarantee that the associated truncated higher-order constitutive relations are elliptic (*i.e.* that the related higher-order effective tensor derived from this asymptotic analysis is positive definite). This is a source of potential difficulties in a direct (non-perturbative) numerical solution of the higher-order equations, for example using finite elements. At the same time, the loss of ellipticity may mean that the "truncated" problem is not well-posed (has no solution or lacks uniqueness), or even if it is well-posed, the solution may cease to be a good approximation to the actual solution.

With the aim of resolving these difficulties while remaining at the rigorous side, we develop an approach based on a *combination* of the asymptotic and variational methods. This is expected to provide a more realistic approximation for the actual solution for "not too small" ε. The approach uses the asymptotic techniques from the above cited mathematical literature (mainly, from Bakhvalov & Panasenko [11]),

[1] Triantafyllidis and Bardenhagen [53] have developed their approach to study a more difficult problem of *stability* of periodic solids (still in a non-rigorous fashion).

supplemented by variational considerations. The approach we adopt is somewhat reminiscent of the paper by Vogelius and Papanicolaou [54], where the authors use a similar "projection method" for construction of optimal first-order correctors in homogenisation of problems with non-smooth boundaries. It allows us to construct in a rigorous way higher-order homogenised equations which *are* elliptic and to show that their solutions are closest to the actual solution in a certain variational sense. The structure of the variationally derived higher-order homogenised constitutive relations (Section 1.3) is in agreement with those proposed by phenomenological strain gradient theories (see *e.g.* Toupin [52], Mindlin [33], Fleck & Hutchinson [23]).

A major difficulty one faces in giving a rigorous meaning to the higher-order homogenised equations is in giving the precise sense to the fact that the actual (rapidly oscillating) solution $u^\varepsilon(\boldsymbol{x})$ oscillates about its mean $v(\boldsymbol{x},\varepsilon)$, which is slightly different from the conventional homogenised solution (the weak limit) $v_0(\boldsymbol{x})$. We show that this difficulty can be resolved by resorting to *energy* considerations: the overall energy $I(\varepsilon)$ not only converges to the homogenised energy I_0 determined by $v_0(\boldsymbol{x})$ when ε tends to zero, but also has the *asymptotics* with respect to the small parameter ε whose main term coincides with I_0 and whose higher-order terms are determined by the higher-order terms in the homogenised solution $v(\boldsymbol{x},\varepsilon)$.[2] In this sense, the rapid oscillations of the real field are of a local nature, *i.e.* they do not affect the overall energy even up to the higher orders in ε. From this point of view, the higher-order homogenised equation is a better approximation since the energy of its solution provides a smaller error approximation for the energy of the actual solution.

Another way of cancelling the effect of the oscillations up to higher orders is to consider averaging of solution for a *family* of the boundary value problems constructed from a given one via all possible translations of the periodic medium (this appears to be a simple random medium with the probability measure coinciding with the usual measure on the unit periodicity cell, each point of which determines a translated realization; J.R. Willis, personal communication). As we show, this does cancel the oscillations up to all orders in ε and constructs an "infinite-order homogenised solution", for which the truncation of the formal asymptotic series $v(\boldsymbol{x},\varepsilon)$ (see (1.7)) serves as a rigorous asymptotics.

The energy based variational considerations allow us further to construct in a natural way the higher-order homogenised variational problem whose minimizer solves the Euler-Lagrange equation, which can be called the higher-order homogenised equation. Importantly, these constructions are supplemented by the variational remainder esti-

[2] The asymptotics for the energy $I(\varepsilon)$ we derive is new to the best of our knowledge. However, Bakhvalov and Èglit [10] previously developed a formal procedure for finding higher-order correctors to I_0 in the case of an arbitrary nonlinear energy functional and showed its relation to the general formal algorithm for constructing the infinite-order homogenised equation presented in Bakhvalov & Panasenko [11]. Their approach did not provide rigorous remainder estimates and we hope that our result on rigorous asymptotics for $I(\varepsilon)$ in the case of quadratic functional will serve better understanding of the related issue.

mates establishing that such choice of the higher-order solution is best possible in a certain rigorous variational sense, and that the associated solutions are close to the real solutions.

This establishes the variational construction of the higher-order homogenised equations, which is further compared with the purely asymptotic construction (*cf.* Boutin [15]). The implications for the adequate choice of the higher-order stress-strain relations and for possible numerical implementation are also discussed.

1.1 The problem and asymptotic expansion of its solution

Although the following analysis can be routinely extended to an arbitrary three-dimensional periodic case, we consider in this chapter the deformation of a "two-dimensional" periodic elastic composite (*i.e.*, a composite whose elastic modulus does not depend on the "anti-plane" coordinate) in anti-plane shear. This simplifies the technical details and therefore, we hope, makes the underlying ideas more transparent.

This preparatory section describes the problem under consideration, constructs the full asymptotic expansion of its solution in the standard form and introduces the homogenised differential operator of infinite order following Bakhvalov & Panasenko [11].

1.1.1 Formulation

The problem in hand is that of anti-plane deformation of a periodic elastic medium whose periodicity cell has a small size ε, in the presence of a body force f of a large fixed period of size T.

To formulate the problem mathematically, it is conventional to introduce first a reference two-dimensional periodic elastic medium characterized by the elasticity tensor $\boldsymbol{C}$ whose periodicity cell Q is the square of the unit size: $Q = [0,1]^2$. The components C_{ijpq}, $i, j, p, q = 1, 2, 3$ of the tensor $\boldsymbol{C}$ are assumed to depend on the in-plane coordinates $\boldsymbol{x} = (x_1, x_2)$ but not to depend on the anti-plane coordinate x_3. In the present section the function $\boldsymbol{C}(\boldsymbol{x})$ is assumed smooth, although the analysis can in principle be extended to piecewise smooth functions (*cf.* Bakhvalov & Panasenko [11]) to include, for example, two-phase media with piecewise constant modulus. The tensor $\boldsymbol{C}$ is symmetric and positive-definite, *i.e.*

$$C_{ijpq} = C_{jipq} = C_{pqij}, \quad \alpha_{ij} C_{ijpq} \alpha_{pq} \geq 0,$$

for any values of i, j, p and q and for any symmetric tensor $\boldsymbol{\alpha}$ with equality in the latter relation only if $\boldsymbol{\alpha} = \boldsymbol{0}$ (summation is applied to repeated suffixes).

As usual for problems of homogenisation, the *actual* elastic medium is assumed to have the period's size characterized by a small parameter ε. Such medium is constructed

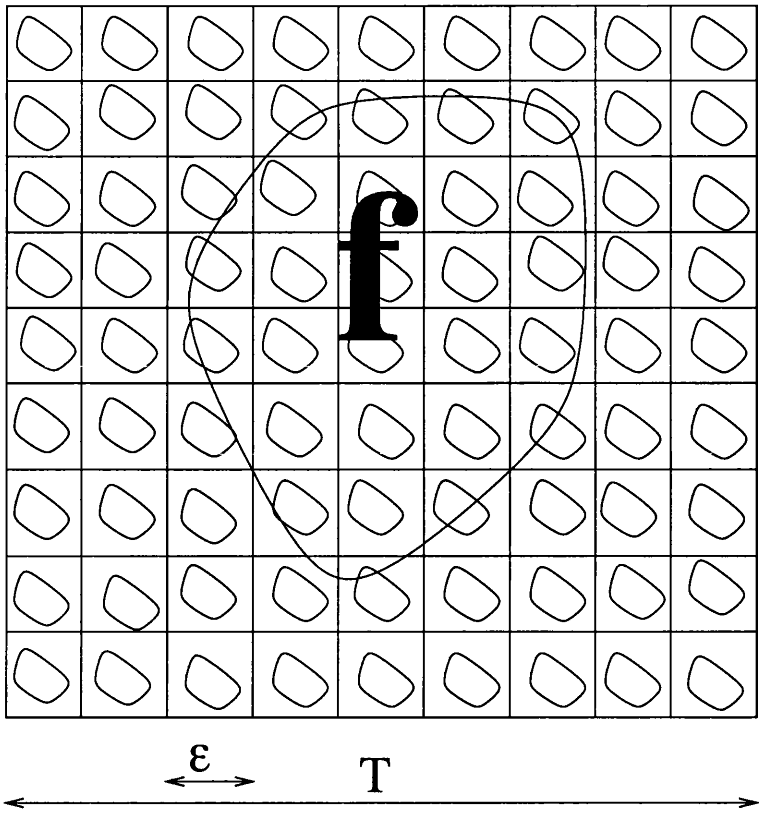

Figure 1.1: An in-plane cross-section of the larger periodicity cell $\mathbf{T}$ of size T consisting of a large number of small periodicity cells of size ε (T/ε is an integer number), in the presence of a $\mathbf{T}$-periodic body force f having zero mean over $\mathbf{T}$.

by re-scaling appropriately the reference periodic medium:

$$C^{\varepsilon}_{ijpq}(\boldsymbol{x}) = C_{ijpq}(\boldsymbol{x}/\varepsilon), \quad \varepsilon > 0.$$

The medium is subjected to a smooth anti-plane body force $\boldsymbol{f}(\boldsymbol{x}) = \Big(f_j(\boldsymbol{x})\Big), j = 1, 2, 3$, where $f_1 = f_2 = 0$ and $f_3 = f$ (see Figure 1.1). The force f is assumed to be periodic with period $\mathbf{T} = [-T, T]^2$ and to have zero mean over $\mathbf{T}$. We assume that T is fixed (independent of ε) and the small variable parameter ε is such that $\varepsilon^{-1}T$ is a large integer number. Provided the elastic tensor satisfies certain symmetry properties, namely $C_{1j3q} = C_{2j3q} = 0$ for $j = 1, 2$, $q = 1, 2, 3$ (satisfied, for example, by isotropic or transversely isotropic media), the body force f initiates the anti-plane deformation characterized by the displacement field $\boldsymbol{u}(\boldsymbol{x}) = \Big(u_j(\boldsymbol{x})\Big)$, $j = 1, 2, 3$ where the in-plane components u_1 and u_2 vanish ($u_1 = u_2 = 0$), and the only non-zero anti-plane component u_3 will be denoted by u.

The resulting equilibrium equation for $\mathbf{T}$-periodic $u(\boldsymbol{x}) = u(x_1, x_2)$[3] reads

$$L_\varepsilon u = -\Big(A_{ij}(\boldsymbol{x}/\varepsilon)u_{,j}\Big)_{,i} = f(\boldsymbol{x}), \quad \int\limits_{\mathbf{T}} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \tag{1.1}$$

where

$$A_{ij} = C_{3i3j},$$

and comma in subscript denotes differentiation. Clearly, the matrix $A(\boldsymbol{y})$ is symmetric and positive definite.

The problem (1.1) admits an equivalent variational formulation: find a function $u(x)$ which minimizes the energy functional $E_\varepsilon(u, f)$ over the set of smooth $\mathbf{T}$-periodic functions:

$$I(\varepsilon, f) = \min_{u(x)} E_\varepsilon(u, f) = \min_{u(x)} \int\limits_{\mathbf{T}} \Big(\frac{1}{2}A_{ij}(\boldsymbol{x}/\varepsilon)u_{,i}u_{,j} - f(\boldsymbol{x})u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}. \tag{1.2}$$

For any positive ε the problem (1.1) (equivalently, (1.2)) has a unique solution $u^\varepsilon(\boldsymbol{x}) = u(\boldsymbol{x}, \varepsilon)$ with zero mean over $\mathbf{T}$:

$$\frac{1}{|\mathbf{T}|}\int\limits_{\mathbf{T}} u^\varepsilon(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$$

(see *e.g.* Bakhvalov and Panasenko [11]). The associated anti-plane strain and stress fields are respectively

$$e^\varepsilon_j(\boldsymbol{x}) = \frac{1}{2}u^\varepsilon_{,j}, \;\; j = 1, 2 \tag{1.3}$$

and

$$\sigma^\varepsilon_i(\boldsymbol{x}) = A_{ij}(\boldsymbol{x}/\varepsilon)u^\varepsilon_{,j}, \;\; i = 1, 2. \tag{1.4}$$

(Here e^ε_j and σ^ε_i stand for $e^\varepsilon_{j3} = e^\varepsilon_{3j}$ and $\sigma^\varepsilon_{i3} = \sigma^\varepsilon_{3i}$, respectively.) We are interested in the behaviour of the solution u^ε when ε is small ($\varepsilon \to 0$), *i.e.*, in the homogenised limit.

1.1.2 Asymptotic expansion of the solution to (1.1).

This subsection follows Bakhvalov and Panasenko [11] in constructing the full asymptotic expansion for u^ε and providing the related error estimates.

First, a formal asymptotic solution of the equation (1.1) is sought in the well-known form separating the slow and the fast variables (see *e.g.* Bensoussan *et al* [13], Sanchez-Palencia [38], Bakhvalov & Panasenko [11] and further references therein):

$$u^\varepsilon(\boldsymbol{x}) \sim \sum_{m=0}^{\infty} \varepsilon^m u_m(\boldsymbol{x}, \boldsymbol{x}/\varepsilon), \tag{1.5}$$

[3]Henceforth in this chapter $\boldsymbol{x}$ denotes the two-dimensional vector (x_1, x_2).

where the functions $u_m(\boldsymbol{x}, \boldsymbol{y})$, $m = 0, 1, 2, \ldots$ are Q-periodic with respect to the fast variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ and $\mathbf{T}$-periodic with respect to the "slow" variable $\boldsymbol{x}$. The idea of the classical ansatz (1.5) is to seek the solution as a decomposition in sequential powers of the small parameter ε whose coefficients, the functions u_l, are periodically oscillating with respect to the fast variable while the oscillation parameters are modulated by the dependency on the slow variable $\boldsymbol{x}$.

It was shown by Bakhvalov and Panasenko [11] along with many others that if the infinite series (1.5) is formally substituted into equation (1.1), the requirement that (1.1) is satisfied to all orders of ε produces a set of differential equations determining sequentially u_0, u_1, u_2, *etc.* Executing this routinely, (which is also summarized in the Appendix A to this chapter) Bakhvalov and Panasenko [11, Chapter 4, Section 2] have shown that the series on the right hand side of (1.5) has the following double-series structure:

$$u^\varepsilon(\boldsymbol{x}) \sim v(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{|k|=l} N_k(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x}, \varepsilon), \tag{1.6}$$

where

$$v(\boldsymbol{x}, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}). \tag{1.7}$$

(In Appendix A we show that in effect the summation in the formula (1.7) applies only to the even values of s.) Here the following conventional notation has been introduced: $k := k_1 \ldots k_l$ is a multi-index, where $k_j = 1, 2$; $j = 1, \ldots, l$; $|k| = l$ is the "length" of k and $D^k v = v_{,k_1 \ldots k_l}$. The summation with respect to $|k| = l$ in (1.6) means the sum of 2^l terms for all possible k of the length l. The "microstructural" functions $N_k(\boldsymbol{y})$ are Q-periodic in $\boldsymbol{y}$ and have zero mean over Q:

$$\langle N_k \rangle := \frac{1}{|Q|} \int_Q N_k(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = 0.$$

(Throughout the text, $\langle \cdot \rangle$ denotes the averaging over the periodicity cell Q. Notice that the factor $1/|Q|$ in front of the integral may be omitted since $|Q| = 1$, which is done henceforth.) The functions $v_s(\boldsymbol{x})$ are $\mathbf{T}$-periodic with zero mean over $\mathbf{T}$, and do not depend on the fast variable $\boldsymbol{x}/\varepsilon$. The double series (1.6)–(1.7) is clearly a particular case of (1.5) where the functions u_m are

$$u_m(\boldsymbol{x}, \boldsymbol{y}) = v_m(\boldsymbol{x}) + \sum_{l=1}^{m} \sum_{|k|=l} N_k(\boldsymbol{y}) D^k v_{m-l}(\boldsymbol{x})$$

with a finite number of terms entering the summation for each m.

The formal asymptotic procedure described in Bakhvalov & Panasenko [11] uniquely defines the functions N_k and v_s entering (1.6), (1.7). This procedure is summarized in Appendix A for the reader's convenience. The representation (1.6)–(1.7) may be viewed as consisting of the "homogenised" (slowly varying) part $v(\boldsymbol{x},\varepsilon)$ and of the rapidly oscillating "tail" corresponding to the summation term in (1.6). The oscillations are described by the "canonical" microstructural functions $N_k(\boldsymbol{y})$ which, importantly, depend only on the microstructure (the functions $A_{ij}(\boldsymbol{y})$) but not on the applied force $f(\boldsymbol{x})$ (see Appendix A). The functions $N_k(\boldsymbol{y})$ can be computed in general by solving numerically the appropriate unit-cell problems. In some cases they can be found analytically, see it e.g. Appendix F for the one-dimensional case (where no macroscopic higher gradients are present) and Section 1.2 for the layered two-dimensional case.

Appendix A shows, in particular, that the formal asymptotic series (1.7) satisfies (formally) the so-called "averaged equation of infinite order" (terminology of Bakhvalov and Panasenko [11])

$$\bar{L}^{(\infty)}v \sim -h_{ij}v_{,ij} - \sum_{l=3}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} h_k D^k v \sim f, \tag{1.8}$$

(In fact, it will be shown in Section 1.2.3 that the summation in (1.8) applies only to the *even* values of l (since h_k satisfy certain anti-symmetry properties when l is odd); in particular, the summation in the right hand side of (1.8) starts in fact from $l = 4$).

The first term of the right-hand side of (1.8)

$$h_{ij} = \langle A_{ij} + A_{ip}N_{j,p} \rangle, \tag{1.9}$$

is a well-known conventional homogenised matrix, and the main term $v_0(\boldsymbol{x})$ in the series (1.7) solves the conventional homogenised equation of second order

$$\bar{L}^{(2)}v_0 = -h_{ij}v_{0,ij}(\boldsymbol{x}) = f(\boldsymbol{x}). \tag{1.10}$$

In (1.9) the Q-periodic functions $N_j(\boldsymbol{y})$ are solutions of the set of well-known unit cell problems:

$$\Big(A_{ip}(\boldsymbol{y})N_{j,p}(\boldsymbol{y})\Big)_{,i} = -A_{jq,q}; \quad \langle N_j \rangle = 0. \tag{1.11}$$

The homogenised matrix (h_{ij}) is symmetric and positive definite, see Appendix A, (*i.e.* the second order operator $\bar{L}^{(2)}$ is elliptic).

Thereby the conventional homogenised tensor (h_{ij}) can be constructed via solution of the unit cell problem (1.11) to determine the functions $N_j(\boldsymbol{y})$ and subsequent averaging according to (1.9), as is well known. Similarly, the higher-order homogenised tensors h_k, $|k| \geq 3$ are found as solutions of the associated higher-order unit cell problems (the equation (1.99) in the Appendix A). For particular periodic media this should be a tractable numerical problem, which will not be discussed here. We concentrate instead on analysis of the rigorous sense of the notion of higher-order homogenised equations and their solutions. Before doing this, we review in the next subsection the technique of remainder estimates establishing that the formal asymptotics (1.6)–(1.7) is a true asymptotics of the actual solution $u^\varepsilon(\boldsymbol{x})$.

1.1.3 Justification (remainder estimates).

In the case of (1.10) the terminology "homogenised equation" is rigorously justified by the fact that for any fixed body force $f(\boldsymbol{x})$, for small ε the actual solution $u^\varepsilon(\boldsymbol{x})$ of the original (inhomogeneous) problem (1.1) is close in a certain sense to $v_0(\boldsymbol{x})$. There exists a vast mathematical literature on giving the precise and most general meaning to this closeness. For our purposes it is sufficient to state that for a fixed f and under all the above assumptions there exists a constant $C(f)$ independent of ε, such that

$$\Big|u^\varepsilon(\boldsymbol{x}) - v_0(\boldsymbol{x})\Big| \le C(f)\varepsilon \tag{1.12}$$

for any $\boldsymbol{x}$. Inequalities of this kind are central in the mathematical theory of homogenisation (see *e.g.* Sanchez-Palencia [38], Jikov *et al* [26]); (1.12) may be interpreted in the sense that the difference between the real solution u^ε and the homogenised solution v_0 is negligible for diminishing ε. Moreover, while the displacements u^ε converge strongly to v_0 when ε tends to zero according to (1.12), the strain fields $e^\varepsilon_j(\boldsymbol{x}) = \frac{1}{2}u^\varepsilon_{,j}$ converge weakly to the homogenised strains $e^0_j = \frac{1}{2}v_{0,j}$, which means that for any smooth test function $\psi(\boldsymbol{x})$

$$\int_{\mathbf{T}} e^\varepsilon_j(\boldsymbol{x})\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \to \int_{\mathbf{T}} e^0_j(\boldsymbol{x})\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

as $\varepsilon \to 0$. The weak convergence language gives precise meaning to the notion that the rapidly oscillating fields $e^\varepsilon_j(\boldsymbol{x})$ oscillate actually about their slowly varying mean $e^0_j(\boldsymbol{x})$.

The above notion of the homogenised equation is further supported by the *convergence of the energy* as follows. In full analogy with (1.2), the solution of the homogenised equation (1.10) minimizes the energy functional $E_0(v, f)$:

$$I_0(f) = \min_{v(x)} E_0(v, f) = \min_{v(x)} \int_{\mathbf{T}} \left(\frac{1}{2}h_{ij}v_{,i}(\boldsymbol{x})v_{,j}(\boldsymbol{x}) - f(\boldsymbol{x})v(\boldsymbol{x})\right)\mathrm{d}\boldsymbol{x} \tag{1.13}$$

over the set of $\mathbf{T}$-periodic functions. It is well known (see *e.g.* the above cited references) that the real energy converges to the homogenised energy when $\varepsilon \to 0$, *i.e.*, for any $f(\boldsymbol{x})$,

$$I(\varepsilon, f) \to I_0(f) \quad \text{when} \quad \varepsilon \to 0. \tag{1.14}$$

Bakhvalov and Panasenko [11, Chapter 4, Section 2] have also established that the formal asymptotic solution (1.6)–(1.7) to the original problem (1.1) provides in fact the true asymptotics to the actual solution u^ε up to all orders of ε in the following rigorous sense. For a fixed positive integer number K consider a truncated version of

(1.6)–(1.7):

$$u^{(K)}(\boldsymbol{x},\varepsilon) = v^{(K)}(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{K} \varepsilon^l \sum_{|k|=l} N_k(\boldsymbol{x}/\varepsilon) D^k v^{(K)}(\boldsymbol{x},\varepsilon), \tag{1.15}$$

where

$$v^{(K)}(\boldsymbol{x},\varepsilon) = \sum_{s=0}^{K} \varepsilon^s v_s(\boldsymbol{x}). \tag{1.16}$$

It can be shown that for any f as above there exists a constant $C^{(K)}(f)$ such that

$$\int_{\mathbf{T}} \left(u^\varepsilon(\boldsymbol{x}) - u^{(K)}(\boldsymbol{x},\varepsilon)\right)^2 \mathrm{d}\boldsymbol{x} \le C^{(K)}(f)\varepsilon^{2K}, \tag{1.17}$$

$$\int_{\mathbf{T}} \left(u^\varepsilon_{,j}(\boldsymbol{x}) - u^{(K)}_{,j}(\boldsymbol{x},\varepsilon)\right)^2 \mathrm{d}\boldsymbol{x} \le C^{(K)}(f)\varepsilon^{2K}, \quad j = 1, 2, \tag{1.18}$$

(*cf.* Bakhvalov & Panasenko [11, Chapter 4, Section 2, Theorem 2], and Appendix B of the present chapter). The inequalities (1.17), (1.18) mean that the formal asymptotic solution (1.6)–(1.7) provides in fact a real asymptotics to the actual solution u^ε with the remainder estimates in the increasing powers of small ε according to (1.17).

This ends this introductory section setting a background for our construction of the homogenised equations of "higher order" in a certain rigorous sense on the basis of the formal "averaged equation of infinite order" (1.8). The underlying idea is to view the asymptotic expansion (1.6) as consisting of the first term $v(\boldsymbol{x},\varepsilon)$ which is a homogenised solution, and the "tail" (the sums in the right hand side of (1.6)), which is rapidly oscillating about zero mean. Notice in passing that a straightforward truncating of (1.8) may be not a good choice for the higher-order homogenised equation since the corresponding (singularly perturbed) operator of the higher order may fail to be elliptic (say, truncating it on $l = 4$ would be good only if the fourth order tensor $h_{k_1k_2k_3k_4}$ were negative definite which is not necessarily the case); for an example of this, see Section 1.2.4. The reason is that the infinite-order equation specified by (1.8) is solved by (1.7) in the perturbative formal asymptotics way (see Appendix A). The loss of ellipticity for truncated equation may mean loss of well-posedness (non-existence or non-uniqueness of solution), or even if the unique solution exists it may not be a good approximation to the actual solution $u^\varepsilon(\boldsymbol{x})$. Note that (in contrast to, for example, phase transformation problems) the loss of ellipticity in this context is "unphysical": the actual solution $u^\varepsilon(\boldsymbol{x})$ remains "well-behaved". This indicates that some care should be applied in executing the above strategy. This is done by making appropriate use of the variational formulation (1.2) in conjunction with the asymptotic considerations.

1.2 Higher-order homogenised equations.

Having at hand the full asymptotic expansion (1.6)–(1.7) for the solution u^ε, in which $v_0(\boldsymbol{x})$ entering the first term of the right hand side of (1.6) via (1.7) is the homogenised solution in the above discussed sense, we wish to extend the notion of the homogenised solution to include higher-order terms with respect to the small parameter ε. For this purpose, use of the *variational* formulation (1.2) has proved to be of certain advantage for us.

The relation (1.14) between the original energy $I(\varepsilon, f)$ and the homogenised energy $I_0(f)$ may be interpreted in the sense that in the very limit of the vanishing ε the effect of the rapid oscillations about the mean $v_0(\boldsymbol{x})$ on the total energy vanishes. To explore the effect of the *higher-order* terms from this point of view, we need to replace the energy relation (1.14) by a more accurate one, delivering the *asymptotics* for $I(\varepsilon, f)$ up to higher orders with respect to small ε rather than simply establishing the limit when $\varepsilon \to 0$. As we will show in the forthcoming subsection, the full asymptotics of $I(\varepsilon, f)$ (up to *all* powers of small ε) will be fully determined by the slowly varying homogenised functions $v_s(\boldsymbol{x})$ (see (1.7)) as well as by the coefficients h_k of the infinite-order homogenised equation (1.8). This appears to be, in our opinion, a fundamental property of energy: the rapid oscillations in the total field (1.6) due to the factors $N_k(\boldsymbol{x}/\varepsilon)$ are of a local nature, *i.e.*, they do not affect the overall energy, even up to the higher orders in ε.

1.2.1 Energy asymptotics

Since we know that the minimum of the right hand side of (1.2) is delivered by the solution u^ε of (1.1) (which is the Euler-Lagrange equation to the variational problem (1.2)), clearly,

$$I(\varepsilon, f) = -\int_{\mathbf{T}} \frac{1}{2} A_{ij}(\boldsymbol{x}/\varepsilon) u^\varepsilon_{,i} u^\varepsilon_{,j} \mathrm{d}\boldsymbol{x} = -\frac{1}{2}\int_{\mathbf{T}} f(\boldsymbol{x}) u^\varepsilon(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{1.19}$$

as follows from multiplying (1.1) with $u = u^\varepsilon$ by u^ε and then integrating by parts.

Notice next that as follows from (1.17) and (1.18),

$$u^\varepsilon(\boldsymbol{x}) = u^{(K)}(\boldsymbol{x}, \varepsilon) + R^{(K)}(\boldsymbol{x}, f, \varepsilon), \tag{1.20}$$

where $u^{(K)}(\boldsymbol{x}, \varepsilon)$ is the truncated asymptotic series (1.15)–(1.16) and the remainder $R^{(K)}$ satisfies

$$\int_{\mathbf{T}} \left(R^{(K)}(\boldsymbol{x}, f, \varepsilon)\right)^2 \mathrm{d}\boldsymbol{x} \leq C^{(K)}(f)\varepsilon^{2K}, \tag{1.21}$$

$$\int_{\mathbf{T}} \left(R^{(K)}_{,j}(\boldsymbol{x}, f, \varepsilon)\right)^2 \mathrm{d}\boldsymbol{x} \leq C^{(K)}(f)\varepsilon^{2K}. \tag{1.22}$$

Substitution of (1.20) into (1.19) results in

$$I(\varepsilon, f) = -\frac{1}{2} \int\limits_{\mathbf{T}} f(\boldsymbol{x}) u^{(K)}(\boldsymbol{x}, \varepsilon) \mathrm{d}\boldsymbol{x} + \tilde{R}^{(K)}(f, \varepsilon), \tag{1.23}$$

where

$$\left| \tilde{R}^{(K)}(f, \varepsilon) \right| \leq \tilde{C}^{(K)}(f) \varepsilon^K \tag{1.24}$$

for some constants $\tilde{C}^{(K)}(f)$ independent of ε.

We argue next that (1.23) remains valid (with possibly different values for the ε-independent constants $\tilde{C}^{(K)}(f)$), if in its right hand side $u^{(K)}$ is replaced by $v^{(K)}$ (see (1.16)), *i.e.*, that the contribution to the energy due to the "tail" (summation term) in the right hand side of (1.15) is asymptotically smaller than *any* power of ε. This may be interpreted in the sense that even up to higher-order terms in ε the rapid oscillations due to $N_k(\boldsymbol{x}/\varepsilon)$ have a local nature and do not affect the overall energy.

Proposition 1. For any infinitely smooth $\mathbf{T}$-periodic function $f(\boldsymbol{x})$ and any K there exists a constant $\hat{C}^{(K)}(f)$ such that

$$I(\varepsilon, f) = -\frac{1}{2} \int\limits_{\mathbf{T}} f(\boldsymbol{x}) v^{(K)}(\boldsymbol{x}, \varepsilon) \mathrm{d}\boldsymbol{x} + \hat{R}^{(K)}(f, \varepsilon), \tag{1.25}$$

and

$$\left| \hat{R}^{(K)}(f, \varepsilon) \right| \leq \hat{C}^{(K)}(f) \varepsilon^K. \tag{1.26}$$

Proof:

The contributions of the terms of the "tail" in the right hand side of (1.15) in the integral in (1.23) are of the form

$$\varepsilon^{l+s} \int\limits_{\mathbf{T}} F_{ks}(\boldsymbol{x}) N_k(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x}, \tag{1.27}$$

where $F_{ks}(\boldsymbol{x}) = -f(\boldsymbol{x}) D^k v_s(\boldsymbol{x})$ is an infinitely smooth $\mathbf{T}$-periodic function. Since $N_k(\boldsymbol{y})$ are infinitely smooth Q-periodic functions with zero mean, the integrals of the form (1.27) decay when ε tends to zero faster than any power of ε (this fact is well-known in the mathematical community and is re-derived in Appendix C):

$$\left| \int\limits_{\mathbf{T}} F_{ks}(\boldsymbol{x}) N_k(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x} \right| \leq C_{ks}^{(K)} \varepsilon^K \tag{1.28}$$

for any K, k and s, for some constants $C_{ks}^{(K)}$. Since for any fixed K there is a finite number of terms of the form (1.27) in (1.23), the proposition follows for $\hat{C}^{(K)}(f)$ taken as appropriate combination of $C_{ks}^{(K)}$ and $\tilde{C}^{(K)}(f)$. $\square$

The Proposition 1 provides rigorous asymptotics to the energy $I(\varepsilon, f)$ in the form:

$$I(\varepsilon, f) \sim I_0(f) + \sum_{s=1}^{\infty} \varepsilon^s I_s(f), \tag{1.29}$$

where

$$I_s(f) := -\frac{1}{2} \int\limits_{\mathbf{T}} f(\boldsymbol{x}) v_s(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \quad s = 0, 1, 2, ...$$

and the remainder estimate upon truncation at $s = K$ is delivered by (1.26). Notice that the main term $I_0(f)$ coincides with the one defined by (1.13). To see this, one needs to repeat the argument which has lead us to (1.19) by noticing that the function $v_0(\boldsymbol{x})$ solving the homogenised equation (1.10) minimizes the functional $E_0(v, f)$.

The formula (1.29) establishes that up to all orders of ε the asymptotics of the energy $I(\varepsilon, f)$ is fully determined by the functions $v_s(\boldsymbol{x})$ entering the formal asymptotic series (1.7), and that the main term $v_0(\boldsymbol{x})$ of this series (the homogenised solution according to (1.10)) delivers the main term $I_0(f)$ to the energy asymptotics. From this perspective, viewing $v_0(\boldsymbol{x})$ as the zeroth order homogenised solution, it is natural to wonder if the higher-order terms in the (so far formal) asymptotic expansion (1.7) could be associated in any rigorous sense with a homogenised equation of a higher order. Before proceeding in this direction, we first establish that there exists a natural candidate for the homogenised solution of the infinite order, *i.e.*, a true function $v(\boldsymbol{x}, \varepsilon)$ for which the right hand side of (1.7) delivers the rigorous asymptotics. The following construction which allows us to do this uses some recent ideas of J. R. Willis (personal communication; see also Luciano & Willis [32]).

1.2.2 Infinite-order homogenised solution

The idea is to achieve cancelling of the effect of rapid oscillations due to the factors $N_k(\boldsymbol{x}/\varepsilon)$ on the overall behaviour by considering the average of solutions for a *family* of the boundary value problems of the form (1.1) with a parameter $\boldsymbol{\zeta}$ belonging to Q, where the right hand side $f(\boldsymbol{x})$ is fixed but the periodic matrix function $\left(A_{ij}(\boldsymbol{y})\right)$ is translated by $\boldsymbol{\zeta}$.

To this end, we consider a family of problems of the form:

$$L_\varepsilon^\zeta u = -\left(A_{ij}^\zeta(\boldsymbol{x}/\varepsilon)u_{,j}\right)_{,i} = -\left(A_{ij}(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta})u_{,j}\right)_{,i} = f(\boldsymbol{x}). \tag{1.30}$$

The case $\boldsymbol{\zeta} = 0$ corresponds to the original problem (1.1), and for any $\boldsymbol{\zeta}$ from Q the problem (1.30) is equivalent to (1.1) with $\left(A_{ij}(\boldsymbol{y})\right)$ replaced by the translated matrix valued periodic function $\left(A_{ij}^\zeta(\boldsymbol{y})\right) = \left(A_{ij}(\boldsymbol{y} + \boldsymbol{\zeta})\right)$.

For any $\boldsymbol{\zeta}$ the problem (1.30) defines a unique solution $u^{\zeta,\varepsilon}(\boldsymbol{x})$. Consider the averaging of this solution with respect to the parameter $\boldsymbol{\zeta}$:

$$\bar{u}^{\varepsilon}(\boldsymbol{x}) := \int_{Q} u^{\zeta,\varepsilon}(\boldsymbol{x}) \mathrm{d}\boldsymbol{\zeta}. \tag{1.31}$$

Note in passing that $\bar{u}^{\varepsilon}$ may be viewed as a simple example of "ensemble averaging" with respect to the statistics of all "translated" realizations of the periodic microstructure which are constructed from a given periodic microstructure $A_{ij}(\boldsymbol{x}/\varepsilon)$ by all possible translations within the periodicity cell. From this point of view the integration in (1.31) is the integration with respect to the probability measure which coincides with the usual Lebesgue measure on Q.

We are arguing that the translation-averaging (1.31) cancels the oscillations due to the factors N_k, and that the series in the right hand side of (1.7) is the true asymptotics for $\bar{u}^{\varepsilon}(\boldsymbol{x})$.

Proposition 2. For a given f, the series in the right hand side of (1.7) delivers a rigorous asymptotics to the function $\bar{u}^{\varepsilon}$ in the following sense. For any K there exists a constant $C'^{(K)}(f)$ such that

$$\int_{\mathbf{T}} \left(\bar{u}^{\varepsilon}(\boldsymbol{x}) - \sum_{s=0}^{K} \varepsilon^{s} v_s(\boldsymbol{x}) \right)^2 \mathrm{d}\boldsymbol{x} \leq C'^{(K)}(f)\varepsilon^{2K}. \tag{1.32}$$

Proof:

Notice first that for any $\boldsymbol{\zeta}$ the function $u^{\zeta,\varepsilon}(\boldsymbol{x})$ has as its rigorous asymptotics the series (1.6)–(1.7) where $N_k(\boldsymbol{y})$ is replaced by $N_k^{\zeta}(\boldsymbol{y}) = N_k(\boldsymbol{y}+\boldsymbol{\zeta})$:

$$u^{\zeta,\varepsilon}(\boldsymbol{x}) \sim v(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{\infty} \varepsilon^{l} \sum_{|k|=l} N_k^{\zeta}(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x},\varepsilon). \tag{1.33}$$

This follows by repeating the derivation which has led to (1.6)–(1.7) (Appendix A) for the translated matrix $A_{ij}^{\zeta}(\boldsymbol{y})$ and noticing that at each stage of the asymptotic construction the functions $N_k(\boldsymbol{y})$ will be replaced by their translated counterparts $N_k^{\zeta}(\boldsymbol{y})$. Moreover, repeating the argument which has lead to the remainder estimate (1.17) shows that

$$\int_{\mathbf{T}} \left(u^{\zeta,\varepsilon}(\boldsymbol{x}) - u^{\zeta,(K)}(\boldsymbol{x},\varepsilon) \right)^2 \mathrm{d}\boldsymbol{x} \leq C^{(K)}(f)\varepsilon^{2K}, \tag{1.34}$$

where the constant $C^{(K)}(f)$ is independent of $\boldsymbol{\zeta}$, and $u^{\zeta,(K)}$ denotes the truncation of the asymptotic series for $u^{\zeta,\varepsilon}$, of the type (1.15):

$$u^{\zeta,(K)}(\boldsymbol{x},\varepsilon) = v^{(K)}(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{K} \varepsilon^{l} \sum_{|k|=l} N_k^{\zeta}(\boldsymbol{x}/\varepsilon) D^k v^{(K)}(\boldsymbol{x},\varepsilon). \tag{1.35}$$

Next,

$$\bar{u}^{\varepsilon}(\boldsymbol{x}) - \sum_{j=0}^{K} \varepsilon^j v_j(\boldsymbol{x}) = \int_Q \left(u^{\zeta,\varepsilon}(\boldsymbol{x}) - u^{\zeta,(K)}(\boldsymbol{x},\varepsilon) \right) \mathrm{d}\boldsymbol{\zeta},$$

since clearly

$$\int_Q N_k^{\zeta}(\boldsymbol{x}/\varepsilon) D^k v^{(K)}(\boldsymbol{x},\varepsilon) \mathrm{d}\boldsymbol{\zeta} = 0$$

for any any K, k and $\boldsymbol{x}$, since $D^k v^{(K)}(\boldsymbol{x},\varepsilon)$ is independent of $\boldsymbol{\zeta}$ and

$$\int_Q N_k^{\zeta}(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{\zeta} = \int_Q N_k(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}) \mathrm{d}\boldsymbol{\zeta} = \langle N_k \rangle = 0.$$

Therefore,

$$\int_{\mathbf{T}} \left(\bar{u}^{\varepsilon}(\boldsymbol{x}) - \sum_{j=0}^{K} \varepsilon^j v_j(\boldsymbol{x}) \right)^2 \mathrm{d}\boldsymbol{x} \le \int_{\mathbf{T}} \left(\int_Q \left| u^{\zeta,\varepsilon}(\boldsymbol{x}) - u^{\zeta,(K)}(\boldsymbol{x},\varepsilon) \right| \mathrm{d}\boldsymbol{\zeta} \right)^2 \mathrm{d}\boldsymbol{x}$$

$$\le C^{(K)}(f) \int_Q \int_{\mathbf{T}} \left(u^{\zeta,\varepsilon}(\boldsymbol{x}) - u^{\zeta,(K)}(\boldsymbol{x},\varepsilon) \right)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta} \le C'^{(K)}(f)\varepsilon^{2K},$$

which proves the Proposition. □

Proposition 2 may be interpreted in the sense that the averaging over the translations $\boldsymbol{\zeta}$ eliminates the rapid oscillations in the solution. The resulting "infinite-order homogenised solution" $\bar{u}^{\varepsilon}(\boldsymbol{x})$ can be identified with function $v(\boldsymbol{x},\varepsilon)$ having the asymptotics (1.7). We know further that the truncation of $v(\boldsymbol{x},\varepsilon)$ on its "zeroth" term $v_0(\boldsymbol{x})$ is the "zero order" homogenised solution of the homogenised equation (1.10) which also solves the "zero order" homogenised variational problem (1.13). We wish to explore next if the higher-order terms in (1.7) may be associated in any way with a higher-order homogenised variational problem akin to (1.13) and with related higher-order homogenised equation. Note again that a straightforward truncation of (1.8) might not be good for this purpose: despite the rigorous remainder estimate for the asymptotics (1.6)–(1.7) (see (1.17), (1.18)), and the Proposition 2, the infinite-order homogenised equation (1.8) remains formal (in the asymptotic sense) and its direct truncation may be no good. To repair this we need to resort further to variational considerations.

1.2.3 Higher-order homogenised variational problems, equations and solutions.

The family of boundary value problems (1.30) labelled by the parameter $\boldsymbol{\zeta}$ has an associated family of equivalent variational formulations akin to (1.2):

$$I^{\zeta}(\varepsilon, f) = \min_{u(x)} E_{\varepsilon}^{\zeta}(u, f) = \min_{u(x)} \int\limits_{\mathbf{T}} \left(\frac{1}{2} A_{ij}^{\zeta}(\boldsymbol{x}/\varepsilon) u_{,i} u_{,j} - f(\boldsymbol{x}) u(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x}. \tag{1.36}$$

Notice that the *asymptotics* of $I^{\zeta}(\varepsilon, f)$ is of the form (1.29), which is independent of $\boldsymbol{\zeta}$ (the energy "does not feel" the *phase* of the oscillations due to the translation $\boldsymbol{\zeta}$). Introduce now the $\boldsymbol{\zeta}$-averaged energy functional:

$$\bar{E}_{\varepsilon}(u, f) := \int\limits_{Q} E_{\varepsilon}^{\zeta}(u, f) \mathrm{d}\boldsymbol{\zeta}. \tag{1.37}$$

It acts upon trial functions $u(\boldsymbol{x}, \boldsymbol{\zeta})$. The idea we are going to exploit here is that the operation of finding the minimum of $E_{\varepsilon}^{\zeta}(u, f)$ and the averaging over all possible translations $\boldsymbol{\zeta}$ commute. To this end we also consider the average of energies of the minimizers of $E_{\varepsilon}^{\zeta}(u, f)$ for all translations $\boldsymbol{\zeta}$ as follows

$$\bar{I}(\varepsilon, f) := \int\limits_{Q} I^{\zeta}(\varepsilon, f) \mathrm{d}\boldsymbol{\zeta}.$$

Then,

$$\bar{I}(\varepsilon, f) = \int\limits_{Q} \left(\min_{u(x)} E_{\varepsilon}^{\zeta}(u, f)\right) \mathrm{d}\boldsymbol{\zeta} = \min_{u(x,\zeta)} \int\limits_{Q} E_{\varepsilon}^{\zeta}(u, f) \mathrm{d}\boldsymbol{\zeta} = \min_{u(x,\zeta)} \bar{E}_{\varepsilon}(u, f). \tag{1.38}$$

Clearly, $\bar{I}(\varepsilon, f)$ still has the asymptotics (1.29). In effect, (1.38) is a particular version of a *statistical* variational principle (as specialized to the present elementary statistics) widely used in mechanics of random heterogeneous media (see *e.g.* Willis [56]).

To motivate the subsequent constructions, observe next that the $\boldsymbol{\zeta}$-averaged variational problem (1.38) has $u(\boldsymbol{x}, \boldsymbol{\zeta}) = u^{\zeta,\varepsilon}(\boldsymbol{x})$ as its minimizer, where $u^{\zeta,\varepsilon}(\boldsymbol{x})$ is the solution of the $\boldsymbol{\zeta}$ parametrized boundary value problem (1.30). (Trivially, for arbitrary $u(\boldsymbol{x}, \boldsymbol{\zeta})$ and $\boldsymbol{\zeta}$ fixed

$$E_{\varepsilon}^{\zeta}\Big(u(\cdot, \boldsymbol{\zeta}), f\Big) \geq E_{\varepsilon}^{\zeta}(u^{\zeta,\varepsilon}, f).$$

Upon integration over $\boldsymbol{\zeta}$ this yields

$$\bar{E}_{\varepsilon}(u, f) \geq \bar{E}_{\varepsilon}(u^{\zeta,\varepsilon}, f)$$

for any $u(\boldsymbol{x}, \boldsymbol{\zeta})$, *i.e.*, $u^{\zeta,\varepsilon}$ is a minimizer.) The rigorous *asymptotics* of this minimizer is as we know given by (1.33). The idea is now to construct a higher-order homogenised variational problem by restricting appropriately the set of trial fields in (1.38) in a way motivated by the form of the truncation (1.35) of the asymptotic series (1.33), for a fixed K. By construction, this is expected (and will be shown rigorously later) to be the best finite order approximation in a certain sense to the infinite-order homogenised solution $v(\boldsymbol{x}, \varepsilon)$.

Higher-order homogenised variational problem

Motivated by (1.35), for any fixed ε and $K \geq 2$ introduce the set $U^{(K)}$ of the trial fields $u(\boldsymbol{x}, \boldsymbol{\zeta})$ as follows:

$$U^{(K)} = \left\{ u(\boldsymbol{x}, \boldsymbol{\zeta}) : u(\boldsymbol{x}, \boldsymbol{\zeta}) = v(\boldsymbol{x}) + \sum_{l=1}^{K-1} \varepsilon^l \sum_{|k|=l} N_k^{\zeta}(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x}) \right\}, \tag{1.39}$$

where $v(\boldsymbol{x})$ is an arbitrary[4] **T**-periodic function and $N_k^{\zeta}(\boldsymbol{y}) = N_k(\boldsymbol{y} + \boldsymbol{\zeta})$ are the microstructural functions entering (1.35).

Consider now the variational problem akin to (1.38) but for the trial fields restricted to $U^{(K)}$:

$$\bar{I}^{(K)}(\varepsilon, f) = \min_{u(x,\zeta) \in U^{(K)}} \bar{E}_\varepsilon(u, f). \tag{1.40}$$

The trial fields in (1.39), (1.40) may be viewed as consisting of the slowly varying function $v(\boldsymbol{x})$ with superimposed rapid oscillations characterized by the microstructural functions $N_k(\boldsymbol{y})$ influenced by the derivatives of $v(\boldsymbol{x})$ up to order $K-1$. Obviously,

$$\bar{I}^{(K)}(\varepsilon, f) \geq \bar{I}(\varepsilon, f) \tag{1.41}$$

since $\bar{I}^{(K)}(\varepsilon, f)$ results from minimizing the same functional as in $\bar{I}(\varepsilon, f)$ but over a restricted set of trial fields (1.39).

Evaluate now $\bar{E}_\varepsilon(u, f)$ for $u \in U^{(K)}$. Straightforward substitution of (1.39) into (1.37), (1.36) results in

$$\bar{E}_\varepsilon(u, f) = \int_{\mathbf{T}} \left[\sum_{l,m=1}^{K} \varepsilon^{l+m-2} \sum_{|k|=l, |n|=m} \frac{1}{2} \tilde{h}_{k;n}^{(K)} D^k v(\boldsymbol{x}, \varepsilon) D^n v(\boldsymbol{x}, \varepsilon) - f(\boldsymbol{x}) v(\boldsymbol{x}, \varepsilon) \right] d\boldsymbol{x}. \tag{1.42}$$

Here the coefficients $\tilde{h}_{k;n}^{(K)}$ are defined as follows.

$$\tilde{h}_{k;n}^{(K)} = \left\langle A_{ij}(N_{k,i} + \delta_{ik_1} N_{k_2 \dots k_l})(N_{n,j} + \delta_{jn_1} N_{n_2 \dots n_m}) \right\rangle \tag{1.43}$$

if $1 < |k| < K$ and $1 < |n| < K$. For $|k| = K$ and/or $|n| = K$ $\tilde{h}_{k;n}^{(K)}$ are still determined by (1.43) where the terms $N_{k,i}$ and/or $N_{n,j}$, respectively, are dropped. For $|k| = 1$

[4] In fact, $v(\boldsymbol{x})$ should be taken from the space $H_{0,per}^K(\mathbf{T})$ of functions which belong to the Sobolev space $H^K(\mathbf{R}^2)$ locally, are **T**-periodic and have zero mean over **T**.

and/or $|n| = 1$ $N_{k_2...k_l}$ and/or $N_{n_2...n_m}$ should be replaced by 1. Note that $\tilde{h}^{(K)}_{k;n}$ are symmetric with respect to k and n, i.e. $\tilde{h}^{(K)}_{k;n} = \tilde{h}^{(K)}_{n;k}$ for any multi-indices k and n.

The relation (1.42) motivates considering the functional

$$E^{(K)}(v, f, \varepsilon) = \int_{\mathbf{T}} \left[\sum_{l,m=1}^{K} \varepsilon^{l+m-2} \sum_{|k|=l,|n|=m} \frac{1}{2} \tilde{h}^{(K)}_{k;n} D^k v(\boldsymbol{x}) D^n v(\boldsymbol{x}) - f(\boldsymbol{x}) v(\boldsymbol{x}) \right] d\boldsymbol{x}. \tag{1.44}$$

(Clearly, $\bar{E}_\varepsilon(u, f) = E^{(K)}(v, f, \varepsilon)$ where $u \in U^{(K)}$ is associated with v (see (1.39)).)[5]

Note that $E^{(K)}$ is a convex functional with respect to v for any f and ε. This follows from its construction via the $\boldsymbol{\zeta}$-averaging of a convex quadratic functional E^{ζ}_{ε}. Consider now the energy minimization problem

$$I^{(K)}(f, \varepsilon) = \min_{v(x) \in H^K_{per}(\mathbf{T})} E^{(K)}(v, f, \varepsilon), \tag{1.45}$$

where $H^K_{per}(\mathbf{T})$ is the closure of the set of infinitely smooth $\mathbf{T}$-periodic functions in the Sobolev space $H^K(\mathbf{T})$. We will call (1.45) the *variational problem of order K*. (Recall that $K \geq 2$.)

Let $v_K(\boldsymbol{x}, \varepsilon)$ be the minimizer of (1.45) with zero mean over the period $\mathbf{T}$[6]. We argue that the above introduced notion of the higher-order homogenised variational problem can be further substantiated by showing that v_K is the best choice of a truncated approximation in a certain variational sense.

Proposition 3. For $K \geq 2$, for any v

$$E^{(K)}(v, f, \varepsilon) \geq E^{(K)}(v_K, f, \varepsilon) \geq \bar{I}(\varepsilon, f). \tag{1.46}$$

Further,

$$\bar{I}(\varepsilon, f) = E^{(K)}(v_K, f, \varepsilon) - \hat{r}^{(K)}(\varepsilon, f), \tag{1.47}$$

where

$$0 \leq \hat{r}^{(K)}(\varepsilon, f) \leq c^{(K)}(f) \varepsilon^{2(K-1)} \tag{1.48}$$

for some constant $c^{(K)}(f)$.

Proof:

The relations (1.46) follow trivially from (1.41) and the fact that v_K is a minimizer of (1.45).

To derive (1.47)–(1.48) observe that substitution of the truncated asymptotic solution $u^{(K-1)}$ (see (1.15)–(1.16)) directly into the energy functional $E_\varepsilon(u, f)$ (see (1.2))

[5]Formally putting $K = \infty$ should correspond to the infinite-order variational problem of Bakhvalov and Èglit [10].

[6]It can be shown (using, for example, Fourier transformation) that if f is smooth then the minimiser of (1.45) exists, is unique up to an arbitrary constant, and is in fact infinitely smooth (see Appendix E).

produces the error in evaluation of $I(\varepsilon, f)$ of the order $\varepsilon^{2(K-1)}$:

$$I(\varepsilon, f) = E_\varepsilon\left(u^{(K-1)}, f\right) - r^{(K)}(\varepsilon, f), \tag{1.49}$$

where

$$0 \leq r^{(K)}(\varepsilon, f) \leq c^{(K)}(f)\varepsilon^{2(K-1)}. \tag{1.50}$$

This routinely follows by substituting (1.20) into (1.2), integration by parts, using (1.1) and (1.22).

Likewise, substitution of the ζ-averaged truncated series $u^{\zeta,(K-1)}$ (see (1.35)) into (1.36), (1.37) produces the error of the same order for $\bar{I}(\varepsilon, f)$:

$$\bar{I}(\varepsilon, f) = \bar{E}_\varepsilon\left(u^{\zeta,(K-1)}, f\right) - \bar{r}^{(K)}(\varepsilon, f), \tag{1.51}$$

$$0 \leq \bar{r}^{(K)}(\varepsilon, f) \leq c^{(K)}(f)\varepsilon^{2(K-1)}. \tag{1.52}$$

Now (1.48) follows from (1.51)–(1.52) and (1.46). □

Proposition 3 justifies the choice of the higher-order homogenised variational formulation in terms of the energy asymptotics. This also allows us to introduce in a natural way a higher-order homogenised equation.

Higher-order homogenised equation

The minimizer v_K $(K \geq 2)$ of the variational formulation (1.45) satisfies the Euler-Lagrange equation for (1.44) as follows:

$$-h_{ij}v_{,ij} - \sum_{l'=3}^{2K} \varepsilon^{l'-2} \sum_{|k'|=l'} h_{k'}^{(K)} D^{k'} v = f, \tag{1.53}$$

where

$$h_{k'}^{(K)} = -\sum_{kn=k', 1\leq|k|,|n|\leq K} \frac{1}{2}\left((-1)^{|k|} + (-1)^{|n|}\right)\tilde{h}_{k;n}^{(K)}. \tag{1.54}$$

In the above formula kn is the multi-index of length $|k|+|n|$ of the form $k_1...k_l n_1...n_m$.

We call the equation (1.53) the *homogenised equation of order* $2K$ and its solution $v_K(\boldsymbol{x}, \varepsilon)$ may accordingly be called the homogenised solution of order K. (Recall that $K \geq 2$.)

Notice that for the coefficients $h_{k'}^{(K)}$ defined via (1.54)

$$h_{[k']}^{(K)} = h_{[k']}, \tag{1.55}$$

as long as $|k'| = l' \leq K-1$, where h_k are as in the infinite order homogenised equation (1.8). (In the above formulas the square brackets denote the symmetrization with respect to the multi-indices k', *i.e.* averaging with respect to all permutations of k'.)

This is due to the fact that the full asymptotic series (1.6)–(1.7) solves (formally) both the original boundary value problem (1.1) and the equivalent variational problem (1.2), which ensures the coincidence of the "key" coefficients in $h^{(K)}_{[k']}$ and $h_{[k']}$ in the homogenised equations.[7] The possible discrepancy for $l' \geq K$ is the effect of the truncation. The relation (1.55) can be derived routinely from (1.54), (1.43), (1.98) and (1.94) (omitted here). As a consequence of (1.55) $h_{[k']} = 0$ if l' is odd (since $h^{(K)}_{k'}$ vanishes if l' is odd as follows from (1.54) and (1.43)). This implies that the summation in (1.8) applies only to even values of l.

Importantly, the above introduced notions of the higher-order homogenised equations and homogenised solutions can be substantiated by rigorous remainder estimates using the present variational framework. Namely, we show below that the solution v_K of the finite order homogenised equation (1.53) approximates the infinite-order homogenised solution $\bar{u}^\varepsilon = v(\boldsymbol{x}, \varepsilon)$ up to higher orders in ε.

Proposition 4: For any given f and $K \geq 2$ there exists a constant $\hat{c}^{(K)}(f)$ such that

$$\int_{\mathbf{T}} \Big(v(\boldsymbol{x},\varepsilon) - v_K(\boldsymbol{x},\varepsilon)\Big)^2 \mathrm{d}\boldsymbol{x} \leq \hat{c}^{(K)}(f)\varepsilon^{2(K-1)}. \tag{1.56}$$

The proof of Proposition 4 is given in Appendix D.

An interpretation of Propositions 3 and 4 is that the order K homogenised solution v_K, while satisfying a singularly perturbed elliptic equation (1.53), is the best approximation of the infinite-order homogenised solution $v(\boldsymbol{x}, \varepsilon)$ in the sense of the energy asymptotics (Proposition 3), and recovers at the same time for sufficiently large K the rigorous asymptotics of $v(\boldsymbol{x}, \varepsilon)$ up to all powers of ε. Note in passing that Proposition 3 ensures that not only are the "averaged" functions v_K and v close, but also the variational approximation u_K (constructed by substituting $v = v_K$ into (1.8)) is close to the actual solution u^ε. This may be given a precise meaning using the above variational bounds, and is omitted here.

1.2.4 An example where direct truncation of the infinite-order homogenised equation is not elliptic.

In this subsection we present an example where straightforward fourth-order truncation of the homogenised equation of infinite order (1.8) is not good, in the sense that such truncation does not deliver an elliptic (*i.e.* well-posed) problem.

Consider an isotropic stratified periodic medium with the direction of the layers along the y_1-axis. The elements of the corresponding constitutive matrix are therefore given by $A_{ij}(\boldsymbol{y}) = a(y_2)\delta_{ij}$, where δ_{ij} is the Kronecker's delta. We assume that the positive function $a(y_2)$ is 1-periodic and infinitely smooth.

[7] Clearly, both in (1.8) and in (1.53) the symmetrization of h_k and $h^{(K)}_{k'}$ does not change the equation, since, *e.g.*, $h_k D^k v = h_{[k]} D^k v$.

In this case, it is not difficult to see that $N_1(\boldsymbol{y}) = 0$ and $N_2(\boldsymbol{y})$ depends on y_2 only and satisfies the following equation

$$\Big(a(y_2)N_2'(y_2)\Big)' = -a'(y_2). \tag{1.57}$$

This equation together with the condition $\langle N_2 \rangle = 0$ gives the formula for the derivative of the function N_2 as follows

$$N_2'(y_2) = \left\langle \Big(a(y_2)\Big)^{-1} \right\rangle_{[0,1]}^{-1} \Big(a(y_2)\Big)^{-1} - 1. \tag{1.58}$$

Henceforth in this subsection, we use the same notation $\langle \cdot \rangle$ for the mean over $[0, 1]$ of functions depending only on one variable as for the mean over the cell of periodicity Q. Also, when the argument of a function is not given it is assumed to be y_2.

Thus, the formulas (1.9) for the coefficients h_{ij} specify that

$$h_{11} = \langle a \rangle; \quad h_{22} = \left\langle a^{-1} \right\rangle^{-1}; \quad h_{ij} = 0 \ \text{ if } i \neq j. \tag{1.59}$$

The expressions (1.59) are well known in the homogenisation community. They show that the homogenised medium is anisotropic, having arithmetic and harmonic means of the function $a(y_2)$ as "conductivities" in the directions parallel and orthogonal to the layers, respectively.

Using the recurrence formulas for the functions N_k (see Appendix A, (1.99)) and (1.59) we get the following equations for the functions $N_{\alpha\beta}(\boldsymbol{y})$, $\alpha, \beta = 1, 2$, which clearly depend only on the variable y_2

$$(aN_{11}')' + a = \langle a \rangle; \tag{1.60}$$

$$(aN_{\alpha\beta}')' = 0, \quad \alpha \neq \beta; \tag{1.61}$$

$$(aN_{22}')' + (aN_2)' + aN_2' + a = \left\langle a^{-1} \right\rangle^{-1}. \tag{1.62}$$

As usual, the functions $N_{\alpha\beta}$ are sought to have zero mean over $[0, 1]$. Equations (1.61) immediately imply $N_{\alpha\beta} = 0$, $\alpha \neq \beta$. Note also that due to the formula (1.58), the equation (1.62) can be rewritten as

$$(aN_{22}')' + (aN_2)' = 0. \tag{1.63}$$

Integrating the last equation we get

$$aN_{22}' + aN_2 = C_{22}, \tag{1.64}$$

where C_{22} is a constant, which has to be found from the condition $\langle N_{22}' \rangle = 0$. Hence, dividing equation (1.64) by a, taking the average and using the fact that $\langle N_2 \rangle = 0$ we get $C_{22} = 0$ and therefore

$$N'_{22} = -N_2. \tag{1.65}$$

Recall that the formulas for the coefficients $h_{\alpha\beta\gamma}$ of the homogenised equation read as follows

$$h_{\alpha\beta\gamma} = \Big\langle A_{\alpha j}(\boldsymbol{y}) N_{\beta\gamma,j}(\boldsymbol{y}) + A_{\alpha\beta}(\boldsymbol{y}) N_\gamma(\boldsymbol{y}) \Big\rangle \tag{1.66}$$

It is easy to see that because the matrix $\Big(A_{ij}(\boldsymbol{y})\Big)$ is diagonal and because the functions $N_{\beta\gamma}(\boldsymbol{y})$ and $N_\gamma(\boldsymbol{y})$ depend only on the variable y_2, the coefficients h_{111}, h_{121}, h_{122}, h_{221}, and h_{212} are all zeros.

From the formula (1.66) and the identity (1.65) we get the following equalities

$$h_{222} = \langle aN'_{22} + aN_2 \rangle = 0.$$

The remaining three-index coefficients h_{211} and h_{112} are found along the following lines.

$$h_{211} = \langle aN'_{11} \rangle = -\langle a' N_{11} \rangle = \Big\langle (aN'_2)' N_{11} \Big\rangle \tag{1.67}$$

$$= -\langle aN_2 N'_{11} \rangle = \Big\langle N_2 (aN_{11})' \Big\rangle = -\langle N_2 a \rangle. \tag{1.68}$$

Here, in (1.67) we integrate by parts and use the expression for $a'(y_2)$ from (1.57), then in (1.68) we integrate by parts once again taking away the derivative from N_2 and use the equation (1.60) for N_{11}. Applying (1.66) for the last time and using the fact that $N_{12} = 0$, we obtain the formula for h_{112} :

$$h_{112} = \langle aN_2 \rangle.$$

Clearly, the only non-zero three-index coefficients h_{211} and h_{112} have sum zero, and therefore the infinite-order homogenised equation does not contain terms with derivatives of third order (*cf.* the discussion in the paragraph following (1.55)).

The equations for the functions $N_{\alpha\beta\gamma}(\boldsymbol{y})$ are as follows (*cf.* (1.99))

$$\Big(A_{ij}(\boldsymbol{y}) N_{\alpha\beta\gamma,j}(\boldsymbol{y})\Big)_{,i} + \Big(A_{i\alpha}(\boldsymbol{y}) N_{\alpha\beta}(\boldsymbol{y})\Big)_{,i} + A_{\alpha j}(\boldsymbol{y}) N_{\beta\gamma,j}(\boldsymbol{y}) + A_{\alpha\beta}(\boldsymbol{y}) N_\gamma(\boldsymbol{y}). \tag{1.69}$$

Taking into account the fact that the elements of the matrix $\Big(A_{ij}(\boldsymbol{y})\Big)$ and the functions $N_k(\boldsymbol{y})$, $|k| = 1, 2$ depend only on the variable y_2, we infer that the functions $N_{\alpha\beta\gamma}(\boldsymbol{y})$ depend only on y_2 and in the equations (1.69) we should only keep terms with derivatives with respect to y_2. Furthermore, it is not difficult to see that due to the matrix (A_{ij}) being diagonal and due to the functions N_1, N_{12}, N_{21} and the coefficients $h_{\alpha\beta\gamma}$, $\alpha\beta\gamma \neq 211, 122$ being zero, the equations for the functions $N_{\alpha\beta\gamma}$, $\alpha\beta\gamma = 212, 121, 221, 111, 122$ have the following form

$$(aN'_{\alpha\beta\gamma})' = 0.$$

Thus, the functions $N_{\alpha\beta\gamma}$, $\alpha\beta\gamma = 212, 121, 221, 111, 122$ are all zeros.

The only non-zero three-index microstructural functions N_{122}, N_{211}, and N_{222} satisfy the following equations

$$(aN'_{112})' + aN_2 = \langle aN_2 \rangle;$$

$$(aN'_{211})' + (aN_{11})' + aN'_{11} = -\langle aN_2 \rangle;$$

$$(aN'_{222})' + (aN_{22})' + aN'_{22} + aN_2 = 0. \tag{1.70}$$

Note that due to the formula (1.65) the equation (1.70) leads to

$$(aN'_{222}) + (aN_{22})' = 0,$$

and therefore acting as with the equation (1.63) we arrive at (*cf.* (1.65))

$$N'_{222} = -N_{22}.$$

The formulas for the four-index coefficients $h_{\alpha\beta\gamma\delta}$ are as follows (*cf.* (1.94)–(1.98))

$$h_{\alpha\beta\gamma\delta} = \langle A_{\alpha j} N_{\beta\gamma\delta,j} + A_{\alpha\beta} N_{\gamma\delta} \rangle.$$

Below we present calculations of each of these coefficients. The machinery employed for obtaining the following formulas is the same as in calculation of the three-index coefficients $h_{\alpha\beta\gamma}$.

$$h_{1211} = h_{2111} = h_{1222} = h_{2122} = h_{2221} = h_{1121}$$

$$= h_{1221} = h_{2121} = h_{1112} = h_{1112} = h_{1212} = 0;$$

$$h_{1111} = \langle aN_{11} \rangle; \tag{1.71}$$

$$h_{2222} = \langle aN'_{222} + aN_{22} \rangle = 0;$$

$$h_{2211} = \langle aN'_{211} + aN_{11} \rangle = \langle -a'N_{211} + aN_{11} \rangle = \Big\langle (aN'_2)'N_{211} + aN_{11} \Big\rangle$$

$$= \langle -aN'_2 N'_{211} + aN_{11} \rangle = \Big\langle N_2 (aN'_{211})' + aN_{11} \Big\rangle$$

$$= \Big\langle -N_2 (aN_{11})' - N_2 aN'_{11} + N_2 h_{211} + aN_{11} \Big\rangle = \langle N'_2 aN_{11} - N_2 aN'_{11} + aN_{11} \rangle$$

$$= \left\langle \left(\left\langle a^{-1} \right\rangle^{-1} a^{-1} - 1 \right) aN_{11} - N_2 aN'_{11} + aN_{11} \right\rangle = -\langle N_2 aN'_{11} \rangle;$$

$$h_{1122} = \langle aN_{22} \rangle;$$

$$h_{2112} = \langle aN'_{112} \rangle = -\langle a'N_{112} \rangle = \Big\langle (aN'_2)'N_{112} \Big\rangle = -\langle aN'_2 N'_{112} \rangle = \Big\langle N_2 (aN'_{112})' \Big\rangle$$

$$= \langle N_2 h_{112} - N_2 aN_2 \rangle = -\langle aN_2^2 \rangle.$$

Thus, the formula for the fourth-order truncation of the infinite-order homogenised equation in the case of a stratified medium is as follows

$$-h_{ij}v_{,ij} - \langle aN_{11}\rangle v_{,1111} - \Big(\langle aN_{22}\rangle - \langle N_2 aN'_{11}\rangle - \langle aN_2^2\rangle\Big)v_{,1122} = f(\boldsymbol{x}). \qquad (1.72)$$

Clearly, the symbol of the equation (1.72) is the following

$$P(\xi_1,\xi_2) = h_{ij}\xi_i\xi_j - \langle aN_{11}\rangle\xi_1^4 + \Big(\langle N_2 aN'_{11}\rangle + \langle aN_2^2\rangle - \langle aN_{22}\rangle\Big)\xi_1^2\xi_2^2, \quad (\xi_1,\xi_2)\in\mathbf{R}^2. \qquad (1.73)$$

Consider the following particular expression for the "conductivity" a:

$$a(y_2) = 2 + \sin(2\pi y_2).$$

Using the formula (1.60) for this choice of a we get

$$N'_{11} = a^{-1}\left(\int_0^{y_2}\Big(\langle a\rangle - a(\tau)\Big)d\tau + N'_{11}(0)\right) = \frac{1}{2+\sin(2\pi y_2)}\left(\int_0^{y_2}\Big(-\sin(2\pi\tau)\Big)d\tau + N'_{11}(0)\right)$$

$$= \frac{1}{2+\sin(2\pi y_2)}\left(\frac{\cos(2\pi y_2)-1}{2\pi} + N'_{11}(0)\right). \qquad (1.74)$$

The unknown value of $N'_{11}(0)$ is found from the condition $\langle N'_{11}\rangle = 0$, which gives $N'_{11}(0) = (2\pi)^{-1}$. Hence the formula (1.74) specifies that

$$N'_{11} = \frac{\cos(2\pi y_2)}{2\pi\Big(2+\sin(2\pi y_2)\Big)}$$

Integrating the last expression we arrive at the following expression for the function N_{11}

$$N_{11} = \frac{1}{4\pi^2}\log\left(1 + \frac{1}{2}\sin(2\pi y_2)\right) + N_{11}(0). \qquad (1.75)$$

The value of $N_{11}(0)$ can be found from the condition $\langle N_{11}\rangle = 0$ but we do not need it.

We substitute the right-hand side of (1.75) into the formula (1.71) for the coefficient h_{1111} to obtain

$$h_{1111} = \langle aN_{11}\rangle = \frac{1}{4\pi^2}\int_0^1 \sin(2\pi\tau)\log\left(1+\frac{1}{2}\sin(2\pi\tau)\right)d\tau = \frac{1}{8\pi^3}\left(\pi - \frac{\pi}{\sqrt{3}} - 1\right) > 0.$$

Hence, it is easy to see that the polynomial $P(\boldsymbol{\xi})$ is not positive definite in the case under consideration:

$$P(\xi_1,0) = \langle a\rangle\xi_1^2 - \langle aN_{11}\rangle\xi_1^4 = 2\xi_1^2 - \frac{1}{8\pi^3}\left(\pi - \frac{\pi}{\sqrt{3}} - 1\right)\xi_1^4 < 0 \quad \text{for large } |\xi_1|.$$

On the other hand, among the coefficients $\tilde{h}^{(2)}_{k;n}$, $|k| = |n| = 2$, which are obtained upon performing variational truncation, there are only two that are non-zero, namely

$$\tilde{h}^{(2)}_{12;12} = \tilde{h}^{(2)}_{22;22} = \langle aN_2^2 \rangle.$$

The corresponding homogenised equation of the fourth order is therefore the following

$$-h_{ij}v_{,ij} + \langle aN_2^2 \rangle v_{,2222} + \langle aN_2^2 \rangle v_{,1122} = f. \tag{1.76}$$

The equation (1.76) is elliptic, with positive definite symbol

$$\tilde{P}(\xi_1, \xi_2) = h_{ij}\xi_i\xi_j + \langle aN_2^2 \rangle \xi_2^4 + \langle aN_2^2 \rangle \xi_1^2\xi_2^2, \quad (\xi_1, \xi_2) \in \mathbf{R}^2.$$

We next aim at discussing briefly the issue of higher-order effective stress-strain relations from the point of view of the present approach.

1.3 Higher-order effective constitutive relations

In an attempt to introduce the higher-order effects into the effective stress-strain relations from the point of view of the present approach, one could either do it on the basis of the asymptotic expansion alone (*cf.* Boutin [15], Triantafyllidis & Bardenhagen [53]) or, alternatively, on the basis of the above variational construction. The remainder estimates will guarantee that in the either case the errors will be small in a certain rigorous sense for sufficiently small ε (although the precise nature of this "smallness" is different in these two approaches, as discussed later). We first summarize below the asymptotic approach and then introduce the rigorous higher-order constitutive relations based on the *variational* approach which is one of the major motivations of the present work.

1.3.1 Asymptotic approach

The asymptotic approach (in its formal asymptotic form) was discussed by Boutin [15] and Triantafyllidis and Bardenhagen [53]. Our construction would modify it as follows, providing also the remainder estimates.

In analogy with the infinite-order homogenised solution $\bar{u}^\varepsilon$ introduced in Section 1.2.2, introduce also the infinite-order homogenised strain and stress as follows.

$$\bar{e}_j^\varepsilon(\boldsymbol{x}) = \int_Q e_j^{\zeta,\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{\zeta}, \;\; j = 1, 2, \tag{1.77}$$

$$\bar{\sigma}_i^\varepsilon(\boldsymbol{x}) = \int_Q \sigma_i^{\zeta,\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{\zeta}, \;\; i = 1, 2 \tag{1.78}$$

with the notation as in Section 1.2.2. Since $e_j^{\zeta,\varepsilon} = \frac{1}{2}u_{,j}^{\zeta,\varepsilon}$, which survives the averaging over ζ,

$$\bar{e}_j^\varepsilon(\boldsymbol{x}) = \frac{1}{2}\bar{u}^\varepsilon_{,j}(\boldsymbol{x}) = \frac{1}{2}v_{,j}(\boldsymbol{x},\varepsilon). \tag{1.79}$$

The asymptotic expansion for $\bar{\sigma}_i^\varepsilon(\boldsymbol{x})$ is derived as follows. Consider the original stress-strain relation,

$$\sigma_i^{\zeta,\varepsilon}(\boldsymbol{x}) = A_{ij}^\zeta(\boldsymbol{x}/\varepsilon)u^{\zeta,\varepsilon}_{,j} = 2A_{ij}^\zeta(\boldsymbol{x}/\varepsilon)e_j^{\zeta,\varepsilon}. \tag{1.80}$$

Substitute into the right hand side of (1.80) the asymptotic expansion (1.33). After differentiation and averaging over ζ according to (1.78) we obtain

$$\bar{\sigma}_i^\varepsilon(\boldsymbol{x}) \sim 2h_{ij}\bar{e}_j^\varepsilon(\boldsymbol{x}) + 2\sum_{l=1}^{\infty}\varepsilon^l\sum_{|k|=l}h_{ikj}D^k\bar{e}_j^\varepsilon(\boldsymbol{x}), \tag{1.81}$$

where h_{ikj} are the coefficients of the infinite-order homogenised equation (1.8) (i and j are indices while k is a multi-index). The relation (1.81) may therefore be viewed as the infinite-order stress-strain relation. It is equivalent to one derived by Boutin [15] and Triantafyllidis and Bardenhagen [53] (when specialized to anti-plane shear). Note that substitution of (1.81), (1.79) into the ζ-averaged equilibrium equation $\bar{\sigma}^\varepsilon_{i,i} = -f$ produces the infinite-order homogenised equation derived by Bakhvalov and Panasenko [11], as expected. Using the technique of truncation and the remainder estimates one could show in a straightforward way that (1.81) gives the rigorous asymptotics for $\bar{\sigma}_i^\varepsilon(\boldsymbol{x})$, *i.e.*, that the difference between actual $\bar{\sigma}^\varepsilon$ and its approximation via the truncation of (1.81) is small for sufficiently small ε.[8]

1.3.2 Variational approach

As an alternative to the above, one could construct higher-order effective relations on the basis of the higher-order variational problem (1.44)–(1.45), or, equivalently, of the higher-order homogenised equation (1.53). This will also provide a small error but in a *variational* rather than asymptotic sense.

To this end considering $v(\boldsymbol{x}) = v_K(\boldsymbol{x})$, $K \geq 2$ (the solution of (1.53)) as a homogenised displacement field, we introduce first the associated strain field

$$e_j(\boldsymbol{x}) = \frac{1}{2}v_{,j}(\boldsymbol{x}). \tag{1.82}$$

[8]It is important to emphasize here that the above truncation is understood *a posteriori*, i.e., assuming that the "exact" averaged solution $\bar{u}^\varepsilon(\boldsymbol{x})$ and therefore $\bar{e}^\varepsilon(\boldsymbol{x})$ are known. An *a priori* truncation of (1.81), *i.e.*, postulating a finite-order stress-strain relation by retaining a finite number of terms in (1.81) with subsequent requirement for $\bar{\sigma}^\varepsilon$ to satisfy the equilibrium may result in the loss of ellipticity for the equation for $\bar{u}$. The *variational* approach is free from this drawback.

Consider the functional (1.44) which could be rewritten in terms of strains as follows:

$$E^{(K)}(v,f,\varepsilon) = \int_{\mathbf{T}} \left[\sum_{l,m=0}^{K-1} \varepsilon^{l+m} \sum_{|k|=l,|n|=m} 2\tilde{h}^{(K)}_{ik;jn} D^k e_i(\boldsymbol{x}) D^n e_j(\boldsymbol{x}) - f(\boldsymbol{x})v(\boldsymbol{x}) \right] \mathrm{d}\boldsymbol{x}. \tag{1.83}$$

Take the variation of the functional (1.83) taking also into account the symmetry of the right hand side:

$$\delta E^{(K)}(v,f,\varepsilon) = \int_{\mathbf{T}} \left[2 \sum_{|n|\le K-1} \sigma_j^{(n)}(\boldsymbol{x}) \delta D^n e_j(\boldsymbol{x}) - f(\boldsymbol{x})\delta v(\boldsymbol{x}) \right] \mathrm{d}\boldsymbol{x}, \tag{1.84}$$

where

$$\sigma_j^{(n)}(\boldsymbol{x}) := 2 \sum_{l=0}^{K-1} \varepsilon^{l+|n|} \sum_{|k|=l} \tilde{h}^{(K)}_{ik;jn} D^k e_i(\boldsymbol{x}) \tag{1.85}$$

play the role of the "work dual" to the strain gradients $D^n e$ and may be called effective higher-order stresses.

It can be checked that the case $|n| = 0$ corresponds in the main order in ε to the conventional effective stress-strain relations for $\sigma = \sigma^{(0)}$ with the second order effective tensor h_{ij}. Further, taking, for example, in (1.85) $K = 2$, the "first order relations" ($|n| = 1$) read

$$\sigma_j^{(n_p)}(\boldsymbol{x}) = 2\varepsilon \tilde{h}^{(K)}_{i;jn_p} e_i(\boldsymbol{x}) + 2\varepsilon^2 \sum_{k_q} \tilde{h}^{(K)}_{ik_q;jn_p} D^{k_q} e_i(\boldsymbol{x}), \tag{1.86}$$

where $\sigma^{(n_p)}$ may be recognized as the "main" higher-order stress (*cf.* Toupin [52], Mindlin [33], Fleck & Hutchinson [23]). Obviously, going into higher orders in K and $|n|$ will define the higher-order versions of the stress.

Integration by parts in (1.84), (1.82) and requirement of the stationarity of the functional (which holds at the solution $v = v_K$) imply:

$$\sigma_{j,j} + \sum_{l=1}^{K-1} (-1)^l \sum_{|m|=l} D^{mj} \sigma_j^{(m)} + f = 0,$$

which plays the part of the higher-order equilibrium equation (*cf.* Toupin [52], Mindlin [33], Fleck & Hutchinson [23]).

This completes the variational construction of the higher-order effective relations. A straightforward extension of the variational remainder estimates techniques of the previous sections will ensure that $e_j(\boldsymbol{x})$ and $\sigma_j(\boldsymbol{x})$ so defined will be close to $\bar{e}_j^{\varepsilon}(\boldsymbol{x})$ and $\bar{\sigma}_j^{\varepsilon}(\boldsymbol{x})$ respectively in a certain rigorous sense. We omit the related details.

Discussion

The work presented in this chapter proposes a combination of asymptotic and variational approaches to construct higher-order constitutive relations for overall behaviour of periodic heterogeneous media. The coefficients of these equations are explicitly related to solutions of higher-order unit cell problems. The techniques of remainder estimates developed in the mathematical literature allow one to show that the solutions of the appropriate higher-order equations are close to the actual solutions in a certain rigorous sense.

The variational asymptotic approach developed in this chapter may be advantageous in comparison with the purely asymptotic approach if the parameter ε is "small but not too small", which corresponds to the practically important situations displaying the so-called scale effects, where the larger scale becomes comparable to the smaller scale. In this case a direct solution of the *variational* higher-order homogenised equation may provide a better approximation to the real solution than the perturbative asymptotic solution in the sequential powers of the small parameter ε. The latter assertion relies on the fact that the variational construction is intrinsically such that it tends to minimize the error in a certain variational sense for final values of ε, while the asymptotic approach simply constructs the perturbations assuming that ε is very small. This could be supported by the error estimates which could in principle be quantified to evaluate the maximal possible error, which is beyond the scope of the present work.

For the purpose of numerical implementation, either approach requires calculation of the higher-order homogenised coefficients h_k, $\tilde{h}^{(K)}_{k;n}$ or $h^{(K)}_{k'}$, (see (1.98), (1.94)–(1.96) and (1.99); (1.43) and (1.54), which depend on "canonical" periodic functions $N_k(\boldsymbol{y})$. The latter depend on the functions $A_{ij}(\boldsymbol{y})$ only via a sequence of the periodic cell boundary value problems. These boundary value problems (allowing an equivalent variational formulation) have ultimately to be solved numerically, or, alternatively, one could seek further approximations for these solutions (*e.g.* in the variational formulation, by using the Hashin-Shtrikman type approximations, *cf.* Drugan & Willis [21], Luciano & Willis [31], [32]).

Finally, note that the described higher order effects are absent in one dimension as shown in Appendix F.

Appendix A: Formal asymptotic expansions

In appendices A and B we follow Bakhvalov and Panasenko [11].

Our main objective here is to construct an asymptotic expansion of the solution to the equation

$$L_\varepsilon u = -\Big(A_{ij}(\boldsymbol{x}/\varepsilon)u_{,j}\Big)_{,i} = f(\boldsymbol{x}), \tag{1.87}$$

where the coefficients A_{ij} are Q-periodic smooth functions satisfying the following conditions

$$A_{ij}(\boldsymbol{y}) = A_{ji}(\boldsymbol{y}),$$
$$A_{ij}(\boldsymbol{y})\eta_i\eta_j \geq \nu\eta_i\eta_i \quad \text{for any vector} \quad (\eta_1, \eta_2) \in \mathbf{R}^2, \tag{1.88}$$

where $\nu > 0$ is a constant.

We assume also that $f(\boldsymbol{x})$ is a smooth $\mathbf{T}$-periodic function with zero mean over $\mathbf{T}$.

The proposed formal asymptotic expansion in the notation of Section 2 is

$$u^{\varepsilon}(\boldsymbol{x}) \sim v(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{|k|=l} N_k(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x}, \varepsilon), \tag{1.89}$$

where $k = k_1...k_l$ are multi-indices and the functions $N_k(\boldsymbol{y})$ are Q-periodic in $\boldsymbol{y}$ with zero mean $\Big(\langle N_k \rangle = 0\Big)$. Substituting (1.89) into the equation (1.87) and assembling the terms with equal powers of ε, after some routine manipulations we get

$$f(\boldsymbol{x}) \sim -\sum_{l=1}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} H_k(\boldsymbol{x}/\varepsilon) D^k v(\boldsymbol{x}, \varepsilon), \tag{1.90}$$

where

$$H_k(\boldsymbol{y}) = (A_{ij}N_{k,j})_{,i} + (A_{ik_1}N_{k_2...k_l})_{,i} + A_{k_1 j}N_{k_2...k_l,j} + A_{k_1k_2}N_{k_3...k_l}, \quad |k| \geq 3, \tag{1.91}$$

$$H_k(\boldsymbol{y}) = (A_{ij}N_{k,j})_{,i} + (A_{ik_1}N_{k_2...k_l})_{,i} + A_{k_1 j}N_{k_2...k_l,j} + A_{k_1k_2}, \quad |k| = 2, \tag{1.92}$$

$$H_{k_1}(\boldsymbol{y}) = (A_{ij}N_{k_1,j})_{,i} + (A_{ik_1})_{,i}, \quad |k| = 1. \tag{1.93}$$

By setting

$$T_k(\boldsymbol{y}) = (A_{ik_1}N_{k_2...k_l})_{,i} + A_{k_1 j}N_{k_2...k_l,j} + A_{k_1k_2}N_{k_3...k_l}, \quad |k| \geq 3, \tag{1.94}$$

$$T_{k_1k_2}(\boldsymbol{y}) = (A_{ik_1}N_{k_2})_{,i} + A_{k_1 j}N_{k_2,j} + A_{k_1k_2}, \quad |k| = 2, \tag{1.95}$$

$$T_{k_1}(\boldsymbol{y}) = A_{ik_1,i}, \quad |k| = 1 \tag{1.96}$$

we can rewrite equations (1.91)–(1.93) in the following way

$$H_k(\boldsymbol{y}) = (A_{ij}N_{k,j})_{,i} + T_k(\boldsymbol{y}).$$

Let us require now that

$$H_k(\boldsymbol{y}) = h_k, \tag{1.97}$$

where h_k are some constants independent of $\boldsymbol{y}$. If we can find N_k and h_k satisfying (1.97) then (1.90) reduces to the problem with constant coefficients for $v(\boldsymbol{x}, \varepsilon)$, which can be solved by standard perturbation methods.

To achieve this, we arrange the unknown functions N_k in such a way that if $|k| > |m|$

then the priority of N_k is higher than the priority of N_m with the functions of the same priority being arranged in an arbitrary way. Note that T_k is defined by functions N_m of the smaller priorities $|m| < |k|$. Because of this, we can use induction to find the functions N_k.

Let us assume that l is a positive integer number and let all the functions N_m with $|m| < l$ be found. Using formulas (1.94)–(1.96) we can calculate functions T_k for all indices k, $|k| = l$.

We set

$$h_k = \langle T_k \rangle. \tag{1.98}$$

Then there exists a unique Q-periodic solution to the problem

$$(A_{ij}N_{k,j})_{,i} + T_k(\boldsymbol{y}) = h_k, \quad \langle N_k \rangle = 0, \tag{1.99}$$

where T_k and h_k are known. Thus, we find all the functions N_k with $|k| = l$. This allows to construct successively all the microstructural functions $N_k(\boldsymbol{y})$ for $|k| = 1, 2, ...$ as solutions of the associated unit cell problems (1.99). Notice that this procedure uniquely constructs N_k from the original periodic functions $A_{ij}(\boldsymbol{y})$ describing the periodic heterogeneity.

Taking into account (1.97), we can rewrite (1.90) in the form

$$f(\boldsymbol{x}) \sim \bar{L}^{(\infty)}v \sim -h_{ij}v_{,ij} - \sum_{l=3}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} h_k D^k v, \tag{1.100}$$

which is called the *averaged equation of infinite order.*

A formal asymptotic solution to the equation (1.100) is sought in the form

$$v(\boldsymbol{x}, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}). \tag{1.101}$$

By substituting the series (1.101) into the equation (1.100) we get the following sequence of equations for determining $v_s(\boldsymbol{x})$:

$$-h_{ij}v_{s,ij}(\boldsymbol{x}) = f_s(\boldsymbol{x}), \ s = 0, 1, 2, ... \tag{1.102}$$

where

$$f_0 = f, \quad f_s = -\sum_{l=3}^{s+2} \sum_{|k|=l} h_k D^k v_{s+2-l}, \ s \geq 1. \tag{1.103}$$

In addition, we require that

$$\frac{1}{|\mathbf{T}|} \int\limits_{\mathbf{T}} v_s(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0. \tag{1.104}$$

We show next that the sequence of equations (1.102) uniquely determines the functions $v_s(\boldsymbol{x})$, $s = 0, 1, 2, \ldots$. To establish this, it is sufficient to show that the matrix $\{h_{ij}\}_{i,j=1}^{2}$ (which is in fact the matrix of the "conventional" effective moduli) is symmetric and positive definite (so that (1.102) is an elliptic equation). We reproduce a derivation of this well-known fact below.

Lemma.

The matrix (h_{ij}) is symmetric and positive definite.

Proof:

The function N_j is Q-periodic and according to (1.93), (1.97) satisfies the equation

$$(A_{pq}N_{j,p})_{,q} = -A_{jq,q},$$

where the functions A_{pq} are Q-periodic as well. Thus, for any Q-periodic function ϕ the following identity holds as a result of integration by parts:

$$\Big\langle (A_{pq}N_{j,p} + A_{jq})\phi_{,q} \Big\rangle = 0.$$

In particular, for $\phi = N_i$ we get

$$\Big\langle (A_{pq}N_{j,p} + A_{jq})N_{i,q} \Big\rangle = 0. \tag{1.105}$$

Hence, taking into account that the matrix (A_{ij}) is symmetric and positive definite, we obtain from (1.91), (1.97) and using (1.105)

$$h_{ij} = \langle A_{ij} + A_{ip}N_{j,p} \rangle = \Big\langle (A_{pq}N_{j,p} + A_{jq})N_{i,q} + A_{pi}N_{j,p} + A_{ji} \Big\rangle$$

$$= \Big\langle A_{pq}(N_{j,p} + \delta_{jp})(N_{i,q} + \delta_{iq}) \Big\rangle.$$

One can easily see from the last equality that the matrix (h_{ij}) is symmetric and positively definite. □

The lemma assures that the sequence of functions v_s that satisfy (1.103) and (1.104) exists and is unique. Note that using the fact that those coefficients h_k for which $|k|$ is odd are zeros (see the discussion after the formula (1.55)), we prove by induction from the formulas (1.102) and (1.103) that the functions f_s and v_s with odd indices are zeros, and therefore the summation in (1.101) and (1.7) applies in fact only to the even values of s.

Appendix B: Justification of the formal asymptotics

Let u^ε be the solution to the equation (1.87), and

$$v^{(K)}(\boldsymbol{x}, \varepsilon) = \sum_{s=0}^{K} \varepsilon^s v_s(\boldsymbol{x}),$$

$$u^{(K)}(\boldsymbol{x},\varepsilon) = v^{(K)}(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{K} \varepsilon^l \sum_{|k|=l} N_k(\boldsymbol{x}/\varepsilon) D^k v^{(K)}(\boldsymbol{x},\varepsilon).$$

Theorem.

For any positive α and any positive integer K there exists a constant $C^{(K)}(f)$ such that the following inequalities hold

$$\left| \int_{\mathbf{T}} u^{(K)}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| \le C^{(K)}(f)\varepsilon^\alpha, \tag{1.106}$$

$$\left\| u^\varepsilon - u^{(K)} \right\|_{W^{1,2}(\mathbf{T})} \le C^{(K)}(f)\varepsilon^K. \tag{1.107}$$

Here,

$$\|w\|^2_{W^{1,2}(\mathbf{T})} = \int_{\mathbf{T}} \left(\left| w(\boldsymbol{x}) \right|^2 + \sum_{i=1}^{2} \left| w_{,i}(\boldsymbol{x}) \right|^2 \right) \mathrm{d}\boldsymbol{x}.$$

Proof:

The inequality (1.106) follows from Appendix C.

By substituting $u^{(K)}$ into the left-hand side of the equation (1.87) one can verify that the function $u^{(K)}$ is a solution of the equation

$$L_\varepsilon u^{(K)} = f + \varepsilon^{K-1}\theta_K(\boldsymbol{x},\varepsilon), \tag{1.108}$$

where $\left|\theta_K(\boldsymbol{x},\varepsilon)\right| \le C_1(f)$ and $C_1(f)$ is independent of ε. Subtracting the equation (1.87) from the equation (1.108) we get

$$L_\varepsilon \left(u^\varepsilon - u^{(K)} \right) = \varepsilon^{K-1}\theta_K(\boldsymbol{x},\varepsilon). \tag{1.109}$$

Multiplying (1.109) by $u^\varepsilon - u^{(K)}$, integrating by parts, using (1.88) and the Poincaré inequality (see *e.g.* Jikov *et al*, 1994), we obtain

$$\nu \left\| \nabla u^\varepsilon - \nabla u^{(K)} \right\|^2_{L^2(\mathbf{T})} \le C_2 \varepsilon^K \|\theta_K\|_{L^2(\mathbf{T})} \left(\left| \int_{\mathbf{T}} \left(u^\varepsilon(\boldsymbol{x}) - u^{(K)}(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} \right| \right.$$

$$\left. + \left\| \nabla u^\varepsilon - \nabla u^{(K)} \right\|_{L^2(\mathbf{T})} \right),$$

for some constant C_2. Hence, taking into account (1.106),

$$\left\| \nabla u^\varepsilon - \nabla u^{(K)} \right\|_{L^2(\mathbf{T})} \le C_3 \varepsilon^{K-1}$$

with appropriate constant C_3. Applying the Poincaré inequality one more time, we arrive at

$$\|u^\varepsilon - u^{(K)}\|_{W^{1,2}(\mathbf{T})} \le C_4^{(K)} \varepsilon^{K-1}. \tag{1.110}$$

Finally, using (1.110) we get

$$\|u^\varepsilon - u^{(K)}\|_{W^{1,2}(\mathbf{T})} \le \|u^\varepsilon - u^{(K+1)}\|_{W^{1,2}(\mathbf{T})} + \|u^{(K+1)} - u^{(K)}\|_{W^{1,2}(\mathbf{T})}$$

$$\le C_4^{(K)}\varepsilon^K + C_5\varepsilon^K = C^{(K)}(f)\varepsilon^K,$$

which is the required estimate (1.107). □

Appendix C: A version of the Riemann-Lebesgue lemma for the smooth periodic case

Theorem.

Assume that a function $M(\boldsymbol{y})$ that belongs to $L^2(Q)$ and has zero mean over Q, is extended by periodicity to $\mathbf{R}^2$ and that $f(\boldsymbol{x}), \boldsymbol{x} \in \mathbf{R}^2$ is a smooth $\mathbf{T}$-periodic function. Then for any K there exists a constant C_K such that

$$\left| \int\limits_{\mathbf{T}} f(\boldsymbol{x}) M(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x} \right| \le C_K \varepsilon^K.$$

Proof:

Let us denote by $Q_\varepsilon^s, s = 1, ..., \left(\frac{T}{\varepsilon}\right)^2$ the cells of periodicity of the function $M(\boldsymbol{x}/\varepsilon)$ in the square $\mathbf{T}$.

It is well known that the Fourier series

$$M_N(\boldsymbol{y}) = \sum_{m_1=0}^{N} \sum_{m_2=0}^{N} c_{m_1,m_2} \exp(im_1y_1 + im_2y_2),$$

where c_{m_1,m_2} are Fourier coefficients of the function $M(\boldsymbol{y})$, converges in $L^2(Q)$ to the function $M(\boldsymbol{y})$. Notice that $c_{0,0} = 0$ since M has zero mean.

Thus, for any s and $\varepsilon > 0$ we get

$$\int\limits_{Q_\varepsilon^s} f(\boldsymbol{x}) M_N(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x} \to \int\limits_{Q_\varepsilon^s} f(\boldsymbol{x}) M(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x} \quad \text{as} \quad N \to \infty.$$

Now, for any fixed $N \in \mathbf{N}$, s, $\varepsilon > 0$, and $r \in \mathbf{N}$ we obtain, employing integration by parts $\left(\text{below } D^{r,s}f \text{ stands for } \frac{\partial^{r+s}f}{\partial x_1^r \partial x_2^s}\right)$:

$$\left| \int\limits_{Q_\varepsilon^s} f(\boldsymbol{x}) M_N(\boldsymbol{x}/\varepsilon) \mathrm{d}\boldsymbol{x} \right| =$$

$$= \left| \sum_{m_1=1}^{N} c_{m_1,0} \int\limits_{Q_\varepsilon^s} f(\boldsymbol{x}) \exp(im_1x_1/\varepsilon) \mathrm{d}\boldsymbol{x} + \sum_{m_2=1}^{N} c_{0,m_2} \int\limits_{Q_\varepsilon^s} f(\boldsymbol{x}) \exp(im_2x_2/\varepsilon) \mathrm{d}\boldsymbol{x} \right.$$

$$+\sum_{m_1=1}^{N}\sum_{m_2=1}^{N} c_{m_1,m_2}\int\limits_{Q_\varepsilon^s} f(\boldsymbol{x})\exp(im_1x_1/\varepsilon+im_2x_2/\varepsilon)\mathrm{d}\boldsymbol{x}\Bigg|$$

$$=\Bigg|\sum_{m_1=1}^{N} c_{m_1,0}\frac{(i\varepsilon)^r}{m_1^r}\int\limits_{Q_\varepsilon^s} D^{r,0}f(\boldsymbol{x})\exp(im_1x_1/\varepsilon)\mathrm{d}\boldsymbol{x}$$

$$+\sum_{m_2=1}^{N} c_{0,m_2}\frac{(i\varepsilon)^r}{m_2^r}\int\limits_{Q_\varepsilon^s} D^{0,r}f(\boldsymbol{x})\exp(im_2x_2/\varepsilon)\mathrm{d}\boldsymbol{x}$$

$$+\sum_{m_1=1}^{N}\sum_{m_2=1}^{N} c_{m_1,m_2}\frac{(i\varepsilon)^{2r}}{m_1^r m_2^r}\int\limits_{Q_\varepsilon^s} D^{r,r}f(\boldsymbol{x})\exp(im_1x_1/\varepsilon+im_2x_2/\varepsilon)\mathrm{d}\boldsymbol{x}\Bigg|$$

$$\leq \varepsilon^r\left(\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c^2_{m_1,m_2}\right)^{\frac{1}{2}}\left(\sum_{m_1=1}^{\infty}\frac{1}{m_1^{2r}}+\sum_{m_2=1}^{\infty}\frac{1}{m_2^{2r}}+\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{1}{m_1^{2r}m_2^{2r}}\right)^{\frac{1}{2}}$$

$$\times\left(\max_{\boldsymbol{x}\in\mathbf{T}}\left|D^{r,r}f(\boldsymbol{x})\right|+\max_{\boldsymbol{x}\in\mathbf{T}}\left|D^{0,r}f(\boldsymbol{x})\right|+\max_{\boldsymbol{x}\in\mathbf{T}}\left|D^{r,0}f(\boldsymbol{x})\right|\right)|Q_\varepsilon^s|\leq C_1(r)\varepsilon^{r+2}.$$

Note that the last estimate is uniform with respect to $N\in\mathbf{N}$. Hence, for any s, $\varepsilon>0$, and $r\in\mathbf{N}$ the following inequality holds

$$\left|\int\limits_{Q_\varepsilon^s} f(\boldsymbol{x})M(\boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x}\right|\leq C_1(r)\varepsilon^{r+2}. \tag{1.111}$$

Finally, using (1.111) we get

$$\left|\int\limits_{\mathbf{T}} f(\boldsymbol{x})M(\boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x}\right|\leq\sum_{s=1}^{\left(\frac{T}{\varepsilon}\right)^2}\left|\int\limits_{Q_\varepsilon^s} f(\boldsymbol{x})M(\boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x}\right|\leq\left(\frac{T}{\varepsilon}\right)^2 C_1(r)\varepsilon^{r+2}\leq C(r)\varepsilon^r.$$

This proves the stated result for $r=K$.

Appendix D: Proof of Proposition 4

Let $u_K^\zeta=u_K^\zeta(\boldsymbol{x},\boldsymbol{\zeta})$ be the function that is associated with v_K by the formula (1.39). Now, if we introduce the function

$$R_K(\boldsymbol{x},\boldsymbol{\zeta})=u^{\zeta,\varepsilon}(\boldsymbol{x},\boldsymbol{\zeta})-u_K^\zeta(\boldsymbol{x},\boldsymbol{\zeta}),$$

then, obviously,

$$E^{(K)}(v_K,f,\varepsilon)=\bar{E}_\varepsilon(u_K^\zeta,f)=\int\limits_{Q} E_\varepsilon^\zeta(u_K^\zeta,f)\mathrm{d}\boldsymbol{\zeta}$$

$$= \int_Q \int_{\mathbf{T}} \left(\frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) u^{\zeta}_{K,i} u^{\zeta}_{K,j} - u^{\zeta}_K f \right) \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}$$

$$= \int_Q \int_{\mathbf{T}} \left(\frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) u^{\zeta,\varepsilon}_{,i} u^{\zeta,\varepsilon}_{,j} - \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) u^{\zeta,\varepsilon}_{,i} R_{K,j} \right.$$

$$\left. - \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} u^{\zeta,\varepsilon}_{,j} + \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} R_{K,j} - u^{\zeta,\varepsilon} f + R_K f \right) \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}$$

$$= \int_Q E^{\zeta}_{\varepsilon}(u^{\zeta,\varepsilon}, f) \mathrm{d}\boldsymbol{\zeta} + \int_Q \int_{\mathbf{T}} \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} R_{K,j} \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}$$

$$\bar{E}_{\varepsilon}(u^{\zeta,\varepsilon}, f) + \int_Q \int_{\mathbf{T}} \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} R_{K,j} \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}$$

$$= \bar{I}(\varepsilon, f) + \int_Q \int_{\mathbf{T}} \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} R_{K,j} \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}. \tag{1.112}$$

We have used above integration by parts and the fact that $u^{\zeta,\varepsilon}$ solves (1.30). Using further the fact that the matrix (A_{ij}) is positive definite we get

$$\int_Q \int_{\mathbf{T}} \frac{1}{2} A^{\zeta}_{ij}(\boldsymbol{x}/\varepsilon) R_{K,i} R_{K,j} \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta} \geq \frac{\nu}{2} \int_Q \int_{\mathbf{T}} \left((R_{K,1})^2 + (R_{K,2})^2 \right) \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{\zeta}$$

$$= \frac{\nu}{2} \int_{\mathbf{T}} \int_Q \left((R_{K,1})^2 + (R_{K,2})^2 \right) \mathrm{d}\boldsymbol{\zeta} \mathrm{d}\boldsymbol{x} \geq \frac{\nu}{2} \int_{\mathbf{T}} \left(\left(\int_Q |R_{K,1}| \mathrm{d}\boldsymbol{\zeta} \right)^2 + \left(\int_Q |R_{K,2}| \mathrm{d}\boldsymbol{\zeta} \right)^2 \right) \mathrm{d}\boldsymbol{x}$$

$$\geq \frac{\nu}{2} \int_{\mathbf{T}} \left(\left| \int_Q R_{K,1} \mathrm{d}\boldsymbol{\zeta} \right|^2 + \left| \int_Q R_{K,2} \mathrm{d}\boldsymbol{\zeta} \right|^2 \right) \mathrm{d}\boldsymbol{x} = \frac{\nu}{2} \sum_{i=1}^{2} \int_{\mathbf{T}} \left(\bar{u}^{\varepsilon}_{,i} - v_{K,i} \right)^2 \mathrm{d}\boldsymbol{x}, \tag{1.113}$$

where ν is an ellipticity constant for the equation (1.1).

Hence, from (1.47), (1.48), (1.112), and (1.113) we get the following estimate

$$\sum_{i=1}^{2} \int_{\mathbf{T}} \left(\bar{u}^{\varepsilon}_{,i} - v_{K,i} \right)^2 \mathrm{d}\boldsymbol{x} \leq \frac{2c^{(K)}(f)}{\nu} \varepsilon^{2(K-1)}$$

Finally, using the Poincaré inequality we obtain the required estimate (1.56).

Appendix E: Existence and uniqueness of the minimiser of the variational problem of order K

Here we prove that there exists a unique (up to an arbitrary constant) solution to the minimisation problem (*cf.* (1.44)–(1.45))

$$\min_{v\in H^K_{per}(\mathbf{T})} E^{(K)}(v,f,\varepsilon), \tag{1.114}$$

where the minimised functional is given by the formula

$$E^{(K)}(v,f,\varepsilon)=\int\limits_{\mathbf{T}}\Big(\sum_{l,m=1}^{K}\varepsilon^{l+m-2}\sum_{|k|=l,|n|=m}\frac{1}{2}\tilde{h}^{(K)}_{k,n}D^k v(\boldsymbol{x})D^n v(\boldsymbol{x})-f(\boldsymbol{x})v(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}.$$

Here we assume that the function f is infinitely smooth.

Recall that according to the procedure described in Section 1.2.3, the energy $E^{(K)}(v,f,\varepsilon)$ can be written in the following way

$$\begin{aligned}
E^{(K)}(v,f,\varepsilon)=&\int\limits_{\mathbf{T}}\int\limits_{Q}\Big(\frac{1}{2}A^{\zeta}_{\alpha\beta}(\boldsymbol{x}/\varepsilon)\Big(v(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)\,D^k v(\boldsymbol{x})\Big)_{,\alpha}\Big(v(\boldsymbol{x})\\
&+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)D^k v(\boldsymbol{x})\Big)_{,\beta}-f(\boldsymbol{x})\Big(v(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)D^k v(\boldsymbol{x})\Big)\Big)\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{x}\\
=&\int\limits_{\mathbf{T}}\int\limits_{Q}\Big(\frac{1}{2}A^{\zeta}_{\alpha,\beta}(\boldsymbol{x}/\varepsilon)\Big(v_{,\alpha}(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N^{\zeta}_k(\boldsymbol{y})}{\partial y_\alpha}\Big|_{\boldsymbol{y}=\frac{\boldsymbol{x}}{\varepsilon}}D^k v(\boldsymbol{x})\\
&+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)D^k v_{,\alpha}(\boldsymbol{x})\Big)\Big(v_{,\beta}(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N^{\zeta}_k(\boldsymbol{y})}{\partial y_\beta}\Big|_{\boldsymbol{y}=\frac{\boldsymbol{x}}{\varepsilon}}D^k v(\boldsymbol{x})\\
&+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)D^k v_{,\beta}(\boldsymbol{x})\Big)-f(\boldsymbol{x})\Big(v(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N^{\zeta}_k(\boldsymbol{x}/\varepsilon)D^k v(\boldsymbol{x})\Big)\Big)\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{x}.
\end{aligned}\tag{1.115}$$

We make the change of variables $\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta}=\boldsymbol{\zeta}'$ in the inner integral in (1.115), retaining the symbol $\boldsymbol{\zeta}$ for the variable of integration, then change the order of integrals, arriving at the following expression for the energy

$$\begin{aligned}
E^{(K)}(v,f,\varepsilon)=&\int\limits_{Q}\int\limits_{\mathbf{T}}\Big(\frac{1}{2}A_{\alpha,\beta}(\boldsymbol{\zeta})\Big(v_{,\alpha}(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial \zeta_\alpha}\Big|_{\zeta=\frac{\boldsymbol{x}}{\varepsilon}}D^k v(\boldsymbol{x})+\\
&\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})D^k v_{,\alpha}(\boldsymbol{x})\Big)\Big(v_{,\beta}(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial \zeta_\beta}\Big|_{\zeta=\frac{\boldsymbol{x}}{\varepsilon}}D^k v(\boldsymbol{x})
\end{aligned}$$

$$+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})D^k v_{,\beta}(\boldsymbol{x})\Big)-f(\boldsymbol{x})\Big(v(\boldsymbol{x})+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})D^k v(\boldsymbol{x})\Big)\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}. \quad (1.116)$$

Using Parseval's formula for the inner integral in the expression (1.116) we obtain the following expression for the energy $E^{(K)}(v,f,\varepsilon)$ in terms of the Fourier coefficients $\hat{f}(\boldsymbol{m})$ and $\hat{v}(\boldsymbol{m})$, $\boldsymbol{m}\in\mathbf{Z}^2$ of the functions $f(\boldsymbol{x})$ and $v(\boldsymbol{x})$

$$E^{(K)}(v,f,\varepsilon)=\int_Q\Big(\frac{1}{2}A_{\alpha,\beta}(\boldsymbol{\zeta})\sum_{\boldsymbol{m}\in\mathbf{Z}^2}\Big(im_\alpha+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial\zeta_\alpha}\Big|_{\boldsymbol{\zeta}=\frac{\boldsymbol{x}}{\varepsilon}}(i\boldsymbol{m})^k$$

$$+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(i\boldsymbol{m})^k im_\alpha\Big)\Big(-im_\beta+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial\zeta_\beta}\Big|_{\boldsymbol{\zeta}=\frac{\boldsymbol{x}}{\varepsilon}}(-i\boldsymbol{m})^k$$

$$+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(-i\boldsymbol{m})^k(-im_\beta)\Big)|\hat{v}(\boldsymbol{m})|^2$$

$$-\hat{f}(\boldsymbol{m})\Big(\overline{\hat{v}(\boldsymbol{m})}+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(-i\boldsymbol{m})^k\overline{\hat{v}(\boldsymbol{m})}\Big)\Big)\mathrm{d}\boldsymbol{\zeta}.$$

In the last formula we use the following natural notation: $\boldsymbol{m}^k = m_{k_1}...m_{k_l}$, where $l=|k|$ is the length of the multi-index k.

Thus, the minimisation problem (1.114) can be written equivalently as a set of minimisation problems for Fourier coefficients $\hat{v}(\boldsymbol{m})$ as follows

$$\min_{\hat{v}(m)}\Big(\frac{1}{2}S(\boldsymbol{m})|\hat{v}(\boldsymbol{m})|^2-\hat{f}(\boldsymbol{m})\overline{\hat{v}(\boldsymbol{m})}\Big),\quad \boldsymbol{m}\in\mathbf{Z}^2\setminus\{\mathbf{0}\}, \quad (1.117)$$

where

$$S(\boldsymbol{m})=\int_Q\Big(A_{\alpha,\beta}(\boldsymbol{\zeta})\Big(im_\alpha+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial\zeta_\alpha}(i\boldsymbol{m})^k+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(i\boldsymbol{m})^k im_\alpha\Big)$$

$$\times\Big(-im_\beta+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial\zeta_\beta}(-i\boldsymbol{m})^k+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(-i\boldsymbol{m})^k(-im_\beta)\Big)\Big)\mathrm{d}\boldsymbol{\zeta}. \quad (1.118)$$

It is easy to see that the minimum in the problem (1.117) is achieved on the value

$$\hat{v}(\boldsymbol{m})=\frac{\hat{f}(\boldsymbol{m})}{S(\boldsymbol{m})}, \quad (1.119)$$

provided $S(\boldsymbol{m})\neq 0$. We estimate the expression (1.118) as follows

$$S(\boldsymbol{m})\geq\nu\int_Q\sum_{\alpha=1}^{2}\Big|im_\alpha+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}\frac{1}{\varepsilon}\frac{\partial N_k(\boldsymbol{\zeta})}{\partial\zeta_\alpha}(i\boldsymbol{m})^k+\sum_{l=1}^{K-1}\varepsilon^l\sum_{|k|=l}N_k(\boldsymbol{\zeta})(i\boldsymbol{m})^k im_\alpha\Big|^2\mathrm{d}\boldsymbol{\zeta}$$

$$\geq \nu \sum_{\alpha=1}^{2} \left| \int_Q \left(im_\alpha + \sum_{l=1}^{K-1} \varepsilon^l \sum_{|k|=l} \frac{1}{\varepsilon} \frac{\partial N_k(\zeta)}{\partial \zeta_\alpha} (i\boldsymbol{m})^k + \sum_{l=1}^{K-1} \varepsilon^l \sum_{|k|=l} N_k(\zeta)(i\boldsymbol{m})^k im_\alpha \right) \mathrm{d}\zeta \right|^2$$

$$= \nu \sum_{\alpha=1}^{2} \left| \int_Q im_\alpha \mathrm{d}\zeta \right|^2 = \nu |\boldsymbol{m}|^2. \tag{1.120}$$

Thus, the formula (1.119) makes sense for any $\boldsymbol{m} \in \mathbf{Z}^2 \setminus \{\mathbf{0}\}$.

If the function $f(\boldsymbol{x})$ is smooth, and hence its Fourier coefficients decay faster than any power of $|\boldsymbol{m}|$ as $|\boldsymbol{m}| \to \infty$ then the estimate (1.120) ensures that the Fourier coefficients (1.119) decay faster than any power of $|\boldsymbol{m}|$ too, and thus the corresponding function $v(\boldsymbol{x})$ (defined uniquely up to an arbitrary additive constant) is infinitely smooth. It satisfies the appropriate Euler-Lagrange equation and is therefore the unique solution to the minimisation problem (1.114).

Appendix F: The absence of higher gradients in one dimension

In this section we show that in the 1D case (when the "matrix" A_{ij} is simply a positive scalar function $a = a(y)$, $y \in [0, 1]$) the homogenised constitutive relation (1.81) does not contain higher gradient terms, *i.e.* it is as follows

$$\bar{\sigma}_i^\varepsilon(\boldsymbol{x}) \sim 2h_{ij}\bar{e}_j^\varepsilon(\boldsymbol{x}),$$

where h_{ij} is the standard homogenised matrix.

Indeed, the microstructural function $N_1(y)$ is found from the following equation (*cf.* (1.57))

$$\left(aN_1'\right)' = -a'.$$

(Henceforth in formulas of this section we omit the argument y.) This equation together with the condition $\langle N_1 \rangle = 0$ gives the formula for the derivative of the function N_1 as follows (*cf.* (1.58))

$$N_1' = \left\langle a^{-1} \right\rangle^{-1} a^{-1} - 1$$

Thus, the formula (1.9) for the coefficients h_{11} specifies as (*cf.* (1.59))

$$h_{11} = \left\langle a^{-1} \right\rangle^{-1}, \tag{1.121}$$

i.e. the constant "conductivity" of the homogenised 1D medium is the harmonic mean of the "conductivity" of the original heterogeneous 1D medium, which is well known.

Using the formula (1.121) we get the following equation for the function $N_{11}(y)$ (*cf.*

(1.62))

$$(aN_{11}')' + (aN_1)' + aN_1' + a = \left\langle a^{-1} \right\rangle^{-1}.$$

As usual, the function N_{11} is sought to have zero mean over $[0,1]$. Note that due to the formula (1.58), the equation (1.62) can be rewritten as

$$(aN_{11}')' + (aN_1)' = 0.$$

Integrating the last equation we get

$$aN_{11}' + aN_1 = C, \tag{1.122}$$

where C is a constant, which has to be found from the condition $\langle N_{21}' \rangle = 0$. Hence, dividing equation (1.122) by a, taking the average and using the fact that $\langle N_1 \rangle = 0$ we get $C = 0$ and therefore (*cf.* (1.65))

$$N_{11}' = -N_1. \tag{1.123}$$

Using the formula (1.98) for the coefficient h_{111} of the homogenised equation of infinite order and the identity (1.123), we arrive at the following equalities

$$h_{111} = \langle aN_{11}' + aN_1 \rangle = 0, \tag{1.124}$$

i.e. the three-index coefficient vanishes. We show by induction that the rest of the homogenised coefficients h_k, $|k| \geq 4$ in the 1D case are zeros as well.

To this end, notice that if for fixed positive integer $n \geq 1$ we have $N_{k1}' = -N_k$ and $h_{k11} = 0$, $|k| \leq n$,[9] then the equation for the coefficient N_k, $|k| = n$

$$(aN_{k11}')' + (aN_{k1})' + aN_{k1}' + aN_k = h_{k11}$$

can be rewritten as follows

$$(aN_{k11}')' + (aN_{k1})' = 0,$$

which gives

$$N_{k11}' + N_{k1} = \frac{C_k}{a}, \tag{1.125}$$

where C_k is a constant. Using the condition $\langle N_{k1} \rangle = 0$ we deduce from (1.125) that

$$N_{k11}' = -N_{k1}.$$

Finally, in the same way as with (1.124) we notice that

$$h_{k111} = \langle aN_{k11}' + aN_{k1} \rangle = 0,$$

[9]The multi-index k in this case is simply a string of unities.

which concludes the induction step.

Note that for any $K \geq 2$ the coefficients $\tilde{h}^{(K)}_{k;n}$ given by formula (1.43) are all zeros too, except for the coefficient $\tilde{h}^{(K)}_{1;1} = \left\langle a(N'_1+1)\right\rangle = h_{11}$, so that all the coefficients $h^{(K)}_{k'}$ (see (1.54)) are zeros and therefore, all "higher-order" homogenised equations (1.53) coincide with the homogenised equation of second order

$$-h_{ij}v_{,ij} = f.$$

Chapter 2

Full asymptotic expansion for solutions of nonlinear periodic rapidly oscillating problems

Introduction

Motivated by size effects, a simplest *linear* model has been considered in the previous chapter. However, the actual environment for the scale effects is often nonlinear (*e.g.* in plasticity, see Fleck *et al.* [24]). To explore the effect of nonlinearity, in this chapter we consider the simplest non-linear problem, namely a quasilinear equation in divergence form as follows

$$-\text{div}\, \boldsymbol{j}\Big(\boldsymbol{x}/\varepsilon, \nabla u^\varepsilon(\boldsymbol{x})\Big) = f(\boldsymbol{x}), \tag{2.1}$$

where the nonlinear function $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e})$ is periodic in $\boldsymbol{y}$ and $\varepsilon > 0$ is a small parameter.

The equation (2.1) generalises the linear case considered in Chapter 1, so through understanding the novel features introduced by the nonlinearity we could get closer to understanding the scale effects in the realistic nonlinear problems. In particular, our hope is that we can use in the future the techniques developed in this chapter (for uniformly elliptic nonlinear problems) in conjunction with the results of Chapter 3 (non-uniformly elliptic linear problems) to treat such challenging problems as microstructure behaviour on approaching the point of loss of stability in plasticity.

In the last three decades, several methods have been developed for passing to the limit as $\varepsilon \to 0$ in nonlinear problems like (2.1). Among them are the theory of Γ-convergence of nonlinear functionals originated in De Giorgi & Franzoni [19]; the theory of H-convergence of monotone operators developed by Murat and Tartar [34]; the two-scale convergence introduced by Nguetseng [35] and later advanced by Allaire [4] for the study of nonlinear problems. Also, Suquet [48] studied homogenisation for problems in plasticity.

Bakhvalov and Panasenko [11] have discussed briefly possible extensions of their linear approach (reviewed in the previous chapter) to the general nonlinear case and the structure of the associated formal asymptotic expansion of the solution, and later on Bakhvalov and Èglit [10] presented a general formal algorithm for constructing the corresponding infinite-order homogenised equation. But the problem of accurate construction, finding precise structure and rigorous justification of the full asymptotic expansion for solutions of nonlinear equations has not been addressed. The aim of this chapter is to fill this gap by performing a complete construction of higher-order terms in the asymptotic expansion for the solution of the problem (2.1), together with its rigorous justification.

In Section 2.2 we show that the solution $u^\varepsilon(\boldsymbol{x})$ to the problem (2.1) has full asymptotic expansion (2.10), (2.11) and the infinite-order homogenised equation has the form (2.21). In particular, the higher-order terms in ε of the nonlinear homogenised equation (2.21) involve higher derivatives (strain gradients) of the homogenised solution $v(\boldsymbol{x})$. This asymptotics is further rigorously justified when the nonlinear function $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e})$ satisfies certain technical conditions (Section 2.3). We also discuss some further developments and prospects (Section 2.4) including applications to non-uniformly elliptic problems.

It is worth emphasizing that the full asymptotic expansion is rigorously justified when the "quadratic" uniform ellipticity condition (2.4) is satisfied. This is necessary for resolving and controlling the effect of the higher-order cell problems, which are *linear*. In some sense, each time when finding a higher-order corrector to the already-constructed leading (nonlinear) terms, we perform linearisation about the leading part of the expansion.

An important question is what happens in the absence of uniform ellipticity. In Chapter 3 we thoroughly discuss the issue of how *non-locality* might arise in considering non-uniformly elliptic linear problems. However, by considering the case of the power-law stored energy function in the present chapter (see Section 2.1.3) we argue that even the techniques of this chapter may still be useful for treatment of some non-uniformly elliptic problems.

We use the notation $\nabla_e, \nabla_x, ..., \mathrm{div}_e, \mathrm{div}_x, ...$ *etc* for corresponding differential operators with respect to the appropriate variables. The powers $(\nabla_e)^l, (\nabla_x)^l, ...$ denote corresponding tensors of l-th order derivatives. We omit the subscript when it is clear from the context with respect to which variable the operator is taken. Also, as is conventional in formulas, when indices repeat summation is implied, every time being carried out over the whole range of the index. Throughout the text, symbols '$\cdot$' and '$\otimes$' denote dot product and tensor product, respectively.

2.1 Formulation of the problem

Consider the following equation

$$-\text{div}\, \boldsymbol{j}\Big(\boldsymbol{x}/\varepsilon, \nabla u^\varepsilon(\boldsymbol{x})\Big) = f(\boldsymbol{x}), \quad \varepsilon > 0. \tag{2.2}$$

Here $\boldsymbol{x} \in \mathbf{R}^d$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e})$ is some nonlinear vector function, periodic in $\boldsymbol{y} \in \mathbf{R}^d$ with the periodicity cell $Q = [0,1]^d$ ($d = 2$ or $d = 3$ in physical applications). For example, this function can be conductivity current density or elastic stress tensor (in geometrically linear elasticity). (In the latter case $\boldsymbol{j}$ is a $d \times d$ matrix.) We will consider the case when the unknown function $u^\varepsilon(\boldsymbol{x})$ is scalar and the function $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e})$ takes values in $\mathbf{R}^d$. Then the function $f(\boldsymbol{x})$ is scalar. We also assume that it is periodic with a fixed period $\mathbf{T} = [-T, T]^d$ where T is a multiple of ε, *i.e.* $\varepsilon^{-1}T \in \mathbf{N}$, and the function $f(\boldsymbol{x})$ has zero mean over $\mathbf{T}$.

In this study, we restrict ourselves to the case when there exists a potential $W = W(\boldsymbol{y}, \boldsymbol{e})$ such that $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{y}, \boldsymbol{e})$. We assume that the function $W = W(\boldsymbol{y}, \boldsymbol{e})$ is infinitely smooth, satisfies a growth condition as follows

$$-A_1 + B_1|\boldsymbol{e}|^p \leq W(\boldsymbol{y}, \boldsymbol{e}) \leq A_2 + B_2|\boldsymbol{e}|^p \;\text{ for any } \boldsymbol{y}, \boldsymbol{e} \in \mathbf{R}^d \tag{2.3}$$

with some positive constants A_1, A_2, B_1, B_2 and $p > 1$. Function $W(\boldsymbol{y}, \boldsymbol{e})$ is required to be convex in $\boldsymbol{e}$. Moreover, for purposes of constructing the full asymptotic expansion we will require that the following inequality holds with some constant $\nu > 0$

$$\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j} \eta_i \eta_j \geq \nu \eta_i \eta_i \tag{2.4}$$

for any $\boldsymbol{y}, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\eta} = (\eta_1, ..., \eta_d) \in \mathbf{R}^d$.

Having fixed $A_1, A_2, B_1, B_2, p, \nu$ and $\mathbf{T}-$periodic function $f \in C^\infty(\mathbf{R}^d) \subset L^{p'}(\mathbf{T})$, $1/p + 1/p' = 1$, with zero mean over $\mathbf{T}$ we consider the following variational problem

$$\min_{u \in W^{1,p}_{0,per}(\mathbf{T})} \int_{\mathbf{T}} \bigg(W\Big(\boldsymbol{x}/\varepsilon, \nabla u(\boldsymbol{x})\Big) - f(\boldsymbol{x})u(\boldsymbol{x})\bigg) \mathrm{d}\boldsymbol{x}, \tag{2.5}$$

where $W^{1,p}_{0,per}(\mathbf{T})$ denotes the space of all $\mathbf{T}$-periodic functions from the Sobolev space $W^{1,p}_{loc}(\mathbf{R}^d)$ having zero mean over $\mathbf{T}$, with the norm being $\|u\|_{W^{1,p}_{0,per}(\mathbf{T})} = \|\nabla u\|_{L^p(\mathbf{T})}$.

The functional $F_\varepsilon[u] = \int_{\mathbf{T}} \Big(W(\boldsymbol{x}/\varepsilon, \nabla u(\boldsymbol{x})) - f(\boldsymbol{x})u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}$ is weakly lower semicontinuous and coercive on $W^{1,p}_{0,per}(\mathbf{T})$. Thus, the problem (2.5) has at least one solution in $W^{1,p}_{0,per}(\mathbf{T})$. Equation (2.2) is the Euler-Lagrange equation for the problem (2.5).

Normally, one also needs to impose some restriction on the function $W = W(\boldsymbol{y}, \boldsymbol{e})$ to make sure that the solution is unique and depends continuously on the right-hand side $-f(\boldsymbol{x})$ of the equation (2.2). This is important for the subsequent asymptotic analysis. A typical restriction for this purpose is the requirement of strong monotonicity of the function $\boldsymbol{j}(\boldsymbol{y},\boldsymbol{e}) = \nabla_e W(\boldsymbol{y},\boldsymbol{e})$ with respect to $\boldsymbol{e}$:

$$\Big(\nabla_e W(\boldsymbol{y},\boldsymbol{e}_1) - \nabla_e W(\boldsymbol{y},\boldsymbol{e}_2)\Big)\cdot\Big(\boldsymbol{e}_1-\boldsymbol{e}_2\Big) \geq \alpha|\boldsymbol{e}_1-\boldsymbol{e}_2|^p, \quad \alpha>0, \tag{2.6}$$

where p is the same as in (2.3), for every $\boldsymbol{y}\in\mathbf{T}$ and all $\boldsymbol{e}_1,\boldsymbol{e}_1\in\mathbf{R}^d$. It is well-known that if (2.6) and (2.3) hold then the solution to the problem (2.5) (equivalently, to the problem (2.2)) is unique and continuously depends on $f\in L^{p'}(\mathbf{T})$. We re-derive this below for the reader's convenience.

Note that a solution $u^\varepsilon(\boldsymbol{x})$ to the problem (2.5) is a stationary point of the functional $F_\varepsilon[u]$, *i.e.* the following identity holds

$$\int_{\mathbf{T}} \nabla_e W(\boldsymbol{x}/\varepsilon,\boldsymbol{e})\Big|_{e=\nabla u^\varepsilon(\boldsymbol{x})}\cdot\nabla\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} f(\boldsymbol{x})\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{2.7}$$

for any function $\phi\in W^{1,p}_{0,per}(\mathbf{T})$.[1] Suppose $u_1^\varepsilon(\boldsymbol{x})$ and $u_2^\varepsilon(\boldsymbol{x})$ are solutions to the problem (2.2) with the right-hand sides $-f_1(\boldsymbol{x})$ and $-f_2(\boldsymbol{x})$ respectively. Then in view of (2.7)

$$\int_{\mathbf{T}}\Big(\nabla_e W(\boldsymbol{x}/\varepsilon,\boldsymbol{e})\Big|_{e=\nabla u_1^\varepsilon(\boldsymbol{x})} - \nabla_e W(\boldsymbol{x}/\varepsilon,\boldsymbol{e})\Big|_{e=\nabla u_2^\varepsilon(\boldsymbol{x})}\Big)\cdot\nabla\Big(u_1^\varepsilon(\boldsymbol{x})-u_2^\varepsilon(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}$$

$$=\int_{\mathbf{T}}\Big(f_1(\boldsymbol{x})-f_2(\boldsymbol{x})\Big)\Big(u_1(\boldsymbol{x})-u_2(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}.$$

Using the inequality (2.6) we obtain

$$\int_{\mathbf{T}}\Big(\nabla_e W(\boldsymbol{x}/\varepsilon,\boldsymbol{e})\Big|_{e=\nabla u_1^\varepsilon(\boldsymbol{x})} - \nabla_e W(\boldsymbol{x}/\varepsilon,\boldsymbol{e})\Big|_{e=\nabla u_2^\varepsilon(\boldsymbol{x})}\Big)\cdot\nabla\Big(u_1^\varepsilon(\boldsymbol{x})-u_2^\varepsilon(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}$$

$$\geq\alpha\int_{\mathbf{T}}\Big|\nabla\Big(u_1^\varepsilon(\boldsymbol{x})-u_2^\varepsilon(\boldsymbol{x})\Big)\Big|^p\mathrm{d}\boldsymbol{x} = \alpha\|u_1^\varepsilon(\boldsymbol{x})-u_2^\varepsilon(\boldsymbol{x})\|^p_{W^{1,p}_{0,per}(\mathbf{T})}.$$

On the other hand, the following inequality holds

$$\Big|\int_{\mathbf{T}}\Big(f_1(\boldsymbol{x})-f_2(\boldsymbol{x})\Big)\Big(u_1(\boldsymbol{x})-u_2(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}\Big| \leq \|f_1-f_2\|_{L^{p'}(\mathbf{T})}\|u_1^\varepsilon-u_2^\varepsilon\|_{L^p(\mathbf{T})}\leq$$

$$\leq c\|f_1-f_2\|_{L^{p'}(\mathbf{T})}\|u_1^\varepsilon-u_2^\varepsilon\|_{W^{1,p}_{0,per}(\mathbf{T})},\quad c>0,$$

where Hölder and Poincaré inequalities have been used. Hence, we conclude that

$$\|u_1^\varepsilon-u_2^\varepsilon\|_{W^{1,p}_{0,per}(\mathbf{T})}\leq\left(\frac{c}{\alpha}\|f_1-f_2\|_{L^{p'}(\mathbf{T})}\right)^{\frac{1}{p-1}}. \tag{2.8}$$

[1]Derivation of (2.7) uses the fact that for a smooth convex function $W(\boldsymbol{y},\boldsymbol{e})$ satisfying (2.3) $|\nabla_e W(\boldsymbol{y},\boldsymbol{e})|\leq A+B|\boldsymbol{e}|^{p-1}$ with some positive constants A and B, see Appendix D.

2.2 Formal asymptotic procedure

Following a sketch in Bakhvalov & Panasenko [11], we are seeking a formal asymptotic expansion of the solution to the problem (2.1) in the following form separating the "slow" and the "fast" variables

$$u^{\varepsilon}(\boldsymbol{x}) \sim \sum_{l=0}^{\infty} \varepsilon^l u_l(\boldsymbol{x}/\varepsilon, \boldsymbol{x}), \tag{2.9}$$

where the functions $u_l(\boldsymbol{y}, \boldsymbol{x})$, $l = 0, 1, 2, ...$ are Q-periodic with respect to the "fast" variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ and $\mathbf{T}$-periodic with respect to the "slow" variable $\boldsymbol{x}$. The idea of the classical "ansatz" (2.9) is to seek the solution as a decomposition in sequential powers of the small parameter ε whose "coefficients", the functions u_l, are periodically oscillating with respect to the fast variable while the oscillation parameters are modulated by the dependency on the slow variable $\boldsymbol{x}$.

Substitution of the ansatz (2.9) into the original equation (2.2) leads us to a more specific structure of the functions $u_l(\boldsymbol{y}, \boldsymbol{x})$. Namely, further we consider the following ansatz

$$u^{\varepsilon}(\boldsymbol{x}) \sim v(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l u_l\Big(\boldsymbol{x}/\varepsilon, \nabla v(\boldsymbol{x}, \varepsilon), \nabla\nabla v(\boldsymbol{x}, \varepsilon), ..., \nabla^l v(\boldsymbol{x}, \varepsilon)\Big), \tag{2.10}$$

where

$$v(\boldsymbol{x}, \varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s(\boldsymbol{x}). \tag{2.11}$$

Functions $u_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big)$ are Q-periodic in $\boldsymbol{y}$ and have zero mean over Q in $\boldsymbol{y}$; functions $v_s(\boldsymbol{x})$ are $\mathbf{T}$-periodic with zero mean over $\mathbf{T}$, and do not depend on the fast variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$.

Now, substitute the series (2.10) into the equation (2.2). After differentiation, formal application of the Taylor formula and another differentiation we end up with a formal asymptotic series in the left-hand side of the equation:

$$-\sum_{l=-1}^{\infty} \varepsilon^l H_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big) = f(\boldsymbol{x}), \tag{2.12}$$

where the functions $H_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big)$ are Q-periodic in $\boldsymbol{y}$. In particular,

$$H_{-1}\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big) = \operatorname{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}) + \nabla_y u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)\Big),$$

$$H_0\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})\Big) = \operatorname{div}_x \boldsymbol{j}\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}) + \nabla_y u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)\Big)$$

$$+\mathrm{div}_y\left(\nabla_e \boldsymbol{j}\left(\boldsymbol{y}, \nabla v(\boldsymbol{x}) + \nabla_y u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)\right)\right.$$

$$\left.\cdot\left(\nabla_x u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big) + \nabla_y u_2\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})\Big)\right)\right).$$

At this point we are going to introduce some conditions on the above defined functions $H_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big)$, which will later prove to be plausible in the sense that they provide us with the way in which we can find an asymptotics of the solution $u^\varepsilon(\boldsymbol{x})$, which can be justified.

First, it is natural to require that the function $H_{-1}\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)$ is identically zero:

$$\mathrm{div}_y \boldsymbol{j}\left(\boldsymbol{y}, \nabla v(\boldsymbol{x}) + \nabla_y u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)\right) = 0 \tag{2.13}$$

This can be viewed as an equation for the function $u_1(\boldsymbol{y}, \boldsymbol{z})$, where $\boldsymbol{z} \in \mathbf{R}^d$ is a parameter:

$$\mathrm{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\Big) = 0. \tag{2.14}$$

By virtue of the conditions formulated in the previous section, the last equation has a Q-periodic solution $u_1(\boldsymbol{y}, \boldsymbol{z})$, which is unique up to an arbitrary constant. We impose the condition $\Big\langle u_1(\boldsymbol{y}, \boldsymbol{z})\Big\rangle = 0$ for any $\boldsymbol{z} \in \mathbf{R}^d$, which provides a unique solution $u_1\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x})\Big)$ to the equation (2.13). Note that the function $v(\boldsymbol{x})$ is still unknown. The function $u_1(\boldsymbol{y}, \boldsymbol{z})$ is the solution of the well-known periodic unit-cell problem in the quasilinear case. It minimizes the functional

$$\int_Q W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u(\boldsymbol{y}, \boldsymbol{z})\Big) \mathrm{d}\boldsymbol{y}. \tag{2.15}$$

It is well-known (see *e.g.* Ladyzhenskaya & Uraltseva [30]) that $u_1(\boldsymbol{y}, \boldsymbol{z})$ is smooth with respect to $\boldsymbol{y}$ as a minimizer of regular functional (2.15). In fact, it can also be shown that under the assumptions on the function $W(\boldsymbol{y}, \boldsymbol{e})$ listed in Section 2.1 the function $u_1(\boldsymbol{y}, \boldsymbol{z})$ is smooth with respect to the *pair* of arguments $\boldsymbol{y}$ and $\boldsymbol{z}$. The derivation is nontrivial and uses the implicit function theorem in functional spaces, see Appendix E.

Proceeding further, we require that the functions $H_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big)$ for $l \geq 0$ do not depend on $\boldsymbol{y}$, *i.e.*,

$$H_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big) = h_l\Big(\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big), \quad l = 0, 1, 2, ... \tag{2.16}$$

for some functions h_l depending on the slow variable *only*. This requirement gives a set of recurrence relations for the functions $u_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big)$, $l \geq 2$.

For example, the condition $H_0\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})\Big) = h_0\Big(\nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})\Big)$ gives us the following equation for the function $u_2\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x})\Big)$:

$$-\mathrm{div}_y\Big(\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})|_{e=\nabla v(x)+\nabla_y u_1(y,\nabla v(x))}\cdot\nabla_y u_2\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}),\nabla\nabla v(\boldsymbol{x})\Big)\Big)$$

$$= -h_0\Big(\nabla v(\boldsymbol{x}),\nabla\nabla v(\boldsymbol{x})\Big) + \mathrm{div}_x \boldsymbol{j}\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}) + \nabla_y u_1\Big(\boldsymbol{y},\nabla v(\boldsymbol{x})\Big)\Big)$$

$$+\mathrm{div}_y\Big(\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})|_{e=\nabla v(x)+\nabla_y u_1(y,\nabla v(x))}\cdot\nabla_x u_1\Big(\boldsymbol{y},\nabla v(\boldsymbol{x})\Big)\Big) \tag{2.17}$$

For better understanding of the structure of the last equation, introduce parameters $\boldsymbol{z}\in\mathbf{R}^d$ and $\boldsymbol{w}\in\mathbf{R}^{d\times d}$ standing for $\nabla v(\boldsymbol{x})$ and $\nabla\nabla v(\boldsymbol{x})$ respectively.

The equation (2.17) takes the following form

$$-\mathrm{div}_y\Big(\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})|_{e=z+\nabla_y u_1(y,z)}\cdot\nabla_y u_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{w})\Big)$$

$$= -h_0(\boldsymbol{z},\boldsymbol{w}) + \nabla_z \boldsymbol{j}\Big(\boldsymbol{y},\boldsymbol{z}+\nabla_y u_1(\boldsymbol{y},\boldsymbol{z})\Big)\cdot\boldsymbol{w}$$

$$+\mathrm{div}_y\Big(\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})|_{e=z+\nabla_y u_1(y,z)}\otimes\nabla_z u_1(\boldsymbol{y},\boldsymbol{z})\Big)\cdot\boldsymbol{w}. \tag{2.18}$$

Note that this equation for $u_2(\boldsymbol{y},\boldsymbol{z},\boldsymbol{w})$ with respect to $\boldsymbol{y}$ with parameters $\boldsymbol{z}$ and $\boldsymbol{w}$ is *linear*. (In effect, in finding higher-order correctors to the leading term of the asymptotics we each time linearize the original equation (2.2).) Further, since by our assumption $\boldsymbol{j}(\boldsymbol{y},\boldsymbol{e}) = \nabla_e W(\boldsymbol{y},\boldsymbol{e})$ and hence $\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e}) = (\nabla_e)^2 W(\boldsymbol{y},\boldsymbol{e})$, in view of (2.4) the equation (2.18) is uniformly elliptic. The smoothness of $u_1(\boldsymbol{y},\boldsymbol{z})$ with respect to $\boldsymbol{y}$ and $\boldsymbol{z}$ ensures the smoothness of the right-hand side of (2.18) and therefore the smoothness of u_2. It is well-known that for solvability of such an equation it is necessary and sufficient that the average with respect to $\boldsymbol{y}$ of its right-hand side is zero. This condition gives us the formula for the function $h_0(\boldsymbol{z},\boldsymbol{w})$ as follows

$$h_0(\boldsymbol{z},\boldsymbol{w}) = \nabla\hat{\boldsymbol{j}}(\boldsymbol{z})\cdot\boldsymbol{w}, \tag{2.19}$$

where $\hat{\boldsymbol{j}}(\boldsymbol{z}) = \Big\langle \boldsymbol{j}\Big(\boldsymbol{y},\boldsymbol{z}+\nabla_y u_1(\boldsymbol{y},\boldsymbol{z})\Big)\Big\rangle$.

Define the function $h_0(\boldsymbol{z},\boldsymbol{w})$ by the formula (2.19). Then there exists a unique solution of the equation (2.18) with zero mean over Q. It could be further shown routinely that in the same fashion the functions $u_l\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}),...,\nabla^l v(\boldsymbol{x})\Big), l = 3,4,...$ can be found. The equations for them will have the same structure

$$-\mathrm{div}_y\Big(\nabla_e \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})|_{e=\nabla v(x)+\nabla_y u_1(y,\nabla v(x))}\cdot\nabla_y u_l\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}),...,\nabla^l v(\boldsymbol{x})\Big)\Big)$$

$$= -h_{l-2}\Big(\nabla v(\boldsymbol{x}),...,\nabla^l v(\boldsymbol{x})\Big) + F_l\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}),...,\nabla^l v(\boldsymbol{x})\Big), \tag{2.20}$$

where the function $F_l\Big(\boldsymbol{y},\nabla v(\boldsymbol{x}),...,\nabla^l v(\boldsymbol{x})\Big)$ can be expressed in terms of the functions

$u_{l'}\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big), l' = 1, 2, ..., l-1$, which are already known.

The equation (2.20) is obviously linear and uniformly elliptic, so the solvability condition for this equation is the following

$$h_{l-2}\Big(\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big) = \Big\langle F_l\Big(\boldsymbol{y}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big)\Big\rangle$$

The last equation defines the function $h_{l-2}\Big(\nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\Big)$.

The function $v(\boldsymbol{x})$ is still unknown.

Now as a result of the above construction we have, from (2.12), (2.16), a formal asymptotic equation in the following form involving only the slow variable $\boldsymbol{x}$

$$-\sum_{l=0}^{\infty} \varepsilon^l h_l\Big(\nabla v(\boldsymbol{x}), \nabla\nabla v(\boldsymbol{x}), ..., \nabla^{l+2} v(\boldsymbol{x})\Big) = f(\boldsymbol{x}). \tag{2.21}$$

This equation can be resolved formally by substituting into it the series (2.11) and performing some formal transformations, namely, a series of differentiations in the arguments of the functions h_l and then expanding slowly varying functions h_l into the Taylor series in powers of ε. In this way we obtain a sequence of equations for the functions $v_s(\boldsymbol{x})$.

The first equation of this sequence is the following

$$-h_0\Big(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x})\Big) = f(\boldsymbol{x}). \tag{2.22}$$

Recall that from (2.19)

$$h_0\Big(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x})\Big) = \operatorname{div} \hat{\boldsymbol{j}}\Big(\nabla v_0(\boldsymbol{x})\Big),$$

where $\hat{\boldsymbol{j}}(\boldsymbol{z}) = \Big\langle \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\Big)\Big\rangle$. We review in the Appendix A the fact that there exists a potential function $\hat{W}(\boldsymbol{z})$ for the equation (2.22) such that

$$\hat{\boldsymbol{j}}(\boldsymbol{z}) = \nabla \hat{W}(\boldsymbol{z}). \tag{2.23}$$

The function $\hat{W}(\boldsymbol{z}) = \Big\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\Big)\Big\rangle$ is the "conventional" effective, or homogenised, energy for nonlinear periodic homogenisation.

Hence, the equation (2.22) reads

$$\operatorname{div}\left(\nabla \hat{W}(\boldsymbol{z})\Big|_{\boldsymbol{z}=\nabla v_0(\boldsymbol{x})}\right) = -f(\boldsymbol{x}) \tag{2.24}$$

and admits the equivalent variational formulation

$$\min_{v(x)\in W^{1,p}_{0,per}(\mathbf{T})} \int\limits_{\mathbf{T}} \Big(\hat{W}\big(\nabla v(\boldsymbol{x})\big) - f(\boldsymbol{x})v(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}. \tag{2.25}$$

The homogenised energy $\hat{W}(\boldsymbol{z})$ inherits all the properties of the function $W(\boldsymbol{y},\boldsymbol{e})$ that are of importance to us. In the Appendix B we show that if (2.3) is satisfied then the similar growth condition is fulfilled for $\hat{W}$:

$$-A_1 + B_1|\boldsymbol{z}|^p \le \hat{W}(\boldsymbol{z}) \le A_2 + B_2|\boldsymbol{z}|^p \quad \text{for any } \boldsymbol{z}\in\mathbf{R}^d, \tag{2.26}$$

and in the Appendix C we verify that the function $\hat{W}(\boldsymbol{z})$ has the property of strong monotonicity:

$$\Big(\nabla\hat{W}(\boldsymbol{z}_1) - \nabla\hat{W}(\boldsymbol{z}_2)\Big)\cdot\Big(\boldsymbol{z}_1 - \boldsymbol{z}_2\Big) \ge \alpha|\boldsymbol{z}_1 - \boldsymbol{z}_2|^p, \tag{2.27}$$

as long as (2.6) holds. Also, we prove in the Appendix A that if (2.4) holds then the similar inequality holds for $\hat{W}$:

$$\frac{\partial^2\hat{W}(\boldsymbol{z})}{\partial z_i\partial z_j}\eta_i\eta_j \ge \nu\eta_i\eta_i \tag{2.28}$$

for any $\boldsymbol{z},\boldsymbol{\eta}\in\mathbf{R}^d$. Note that all the constants $(p, A_1, A_2, B_1, B_2, \alpha, \nu)$ entering (2.26)–(2.28) are the same as in (2.3)–(2.6).

Hence, the equation (2.24) has a solution, which is unique up to an arbitrary constant. We choose the function $v_0(\boldsymbol{x})$ to have zero mean over $\mathbf{T}$, *i.e.* $\int_{\mathbf{T}} v_0(\boldsymbol{x})\mathrm{d}x = 0$. If $\hat{W}(\boldsymbol{z})$ is smooth[2], it can be shown that the minimizer of (2.25) is smooth (see *e.g.* Ladyzhenskaya & Uraltseva [30]).

The second equation of the sequence obtained by substituting the series (2.11) into the formal equation (2.21), *i.e.* the equation for $v_1(\boldsymbol{x})$, is linear and can be written in the following way

$$-\left(\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial z_i}\frac{\partial v_1(\boldsymbol{x})}{\partial x_i} + \frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial w_{ij}}\frac{\partial^2 v_1(\boldsymbol{x})}{\partial x_i\partial x_j}\right)\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)}$$

$$= h_1(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x}), \nabla\nabla\nabla v_0(\boldsymbol{x})). \tag{2.29}$$

Note that

$$h_0(\boldsymbol{z},\boldsymbol{w}) = \nabla\hat{\boldsymbol{\jmath}}(\boldsymbol{z})\cdot\boldsymbol{w},$$

where $\hat{\boldsymbol{\jmath}}(\boldsymbol{z}) = \nabla\hat{W}(\boldsymbol{z})$, and so the following identities hold

$$\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial z_i}\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)} = \frac{\partial}{\partial x_j}\left(\frac{\partial h_0(\boldsymbol{z},\boldsymbol{w})}{\partial w_{ij}}\Bigg|_{z=\nabla v_0(x),w=\nabla\nabla v_0(x)}\right) =$$

[2]Smoothness of the function $\hat{W}(\boldsymbol{z})$ is ensured by smoothness of $u_1(\boldsymbol{y},\boldsymbol{z})$, see the discussion on p.59.

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j}\bigg|_{z=\nabla v_0(x)} \right), \quad i = 1, ..., d.$$

In view of these identities we rewrite the equation (2.29) in divergence form as follows

$$-\frac{\partial}{\partial x_i}\left(\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j}\bigg|_{z=\nabla v_0(x)} \frac{\partial v_1}{\partial x_j} \right) = f_1(\boldsymbol{x}), \tag{2.30}$$

where $f_1(\boldsymbol{x}) = h_1\Big(\nabla v_0(\boldsymbol{x}), \nabla\nabla v_0(\boldsymbol{x}), \nabla\nabla\nabla v_0(\boldsymbol{x})\Big)$. It can be shown that the function $f_1(\boldsymbol{x})$ has zero mean over $\mathbf{T}$. The linear equation (2.30) is uniformly elliptic by virtue of (2.28). It follows that there exists a unique solution $v_1(\boldsymbol{x})$ of (2.30) with zero mean over $\mathbf{T}$.

In the same fashion one can proceed with this recurrent procedure of finding the functions $v_s(\boldsymbol{x}), s = 0, 1, ...$ and see that at the s-th step, $s = 2, 3, ...$ the equation for $v_s(\boldsymbol{x})$ can be obtained, which has the following form akin to (2.30)

$$-\frac{\partial}{\partial x_i}\left(\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j}\bigg|_{z=\nabla v_0(x)} \frac{\partial v_s}{\partial x_j} \right) = f_s(\boldsymbol{x}), \tag{2.31}$$

where the function $f_s(\boldsymbol{x})$ is expressed in terms of the functions $v_0, v_1, ..., v_{s-1}$, which are already known, and has zero mean over $\mathbf{T}$.

The equation (2.31) is linear and uniformly elliptic, thus the solution $v_s(\boldsymbol{x})$ with zero mean over $\mathbf{T}$ does exist and is unique.

This completes the procedure of constructing a formal asymptotic solution for the nonlinear equation (2.1).

2.3 Justification of the formal asymptotics (2.10), (2.11)

The asymptotics (2.10), (2.11) can be justified in the following sense. If we truncate both series (2.10) and (2.11) and substitute the truncation of the second series

$$v^{(K)}(\boldsymbol{x}, \varepsilon) = \sum_{s=0}^{K} \varepsilon^s v_s(\boldsymbol{x}). \tag{2.32}$$

into the truncation of the first one

$$u^{(K)}(\boldsymbol{x}, \varepsilon) = v^{(K)}(\boldsymbol{x}, \varepsilon) + \sum_{l=1}^{K} \varepsilon^l u_l\Big(\boldsymbol{x}/\varepsilon, \nabla v^{(K)}(\boldsymbol{x}, \varepsilon), \nabla\nabla v^{(K)}(\boldsymbol{x}, \varepsilon), ..., \nabla^l v^{(K)}(\boldsymbol{x}, \varepsilon)\Big), \tag{2.33}$$

then the following inequality holds with some constant C_{K-1}

$$\left\| u^\varepsilon(\boldsymbol{x}) - u^{(K)}(\boldsymbol{x}, \varepsilon) \right\|_{W^{1,p}_{0,per}(\mathbf{T})} \leq C_{K-1} \varepsilon^{K-1}. \tag{2.34}$$

We prove the inequality (2.34) in the following way. Substitute the sum (2.33) into the original equation (2.2). Using the well-known formula for the remainder of the Taylor series and formulas for the functions H_l obtained in the previous section we get

$$-\text{div } \boldsymbol{j}\left(\boldsymbol{x}/\varepsilon, \nabla u^{(K)}(\boldsymbol{x})\right)$$

$$= f(\boldsymbol{x}) - \sum_{l=-1}^{K-2} \varepsilon^l H_l\Big(\boldsymbol{x}/\varepsilon, \nabla v^{(K)}(\boldsymbol{x}), \nabla\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})\Big) + \varepsilon^{K-1} R_{K-1}(\boldsymbol{x}/\varepsilon, \boldsymbol{x}),$$

where $R_{K-1}(\boldsymbol{y}, \boldsymbol{x})$ is certain polynomial of $\nabla_x u_1 + \nabla_y u_2, ..., \nabla_x u_{K-1} + \nabla_y u_K$, $\nabla_x u_K$ and of $(\nabla_e)^l W(\boldsymbol{y}, \boldsymbol{e})|_{e=e_l(x)}$ with $1 \le l \le K+1$ and uniformly bounded vector functions $\boldsymbol{e}_l(\boldsymbol{x})$. Since the potential $W(\boldsymbol{y}, \boldsymbol{e})$ is assumed to be infinitely smooth for $\boldsymbol{y} \in Q$ and $\boldsymbol{e} \in \mathbf{R}^d$, the remainder $R_{K-1}(\boldsymbol{y}, \boldsymbol{x})$ is uniformly bounded by some constant $\hat{C}_{K-1}$. Proceeding further, we recall that $H_{-1}\Big(\boldsymbol{y}, \nabla v^{(K)}(\boldsymbol{x})\Big) \equiv 0$ and also in view of (2.16)

$$H_l\Big(\boldsymbol{y}, \nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})\Big) = h_l\Big(\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})\Big), \quad l = 0, 1, 2, ...$$

Thus,

$$-\text{div } \boldsymbol{j}\left(\boldsymbol{x}/\varepsilon, \nabla u^{(K)}(\boldsymbol{x})\right) =$$

$$= -\sum_{l=0}^{K-2} \varepsilon^l h_l\Big(\nabla v^{(K)}(\boldsymbol{x}), \nabla\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})\Big) + \varepsilon^{K-1} R_{K-1}(\boldsymbol{x}/\varepsilon, \boldsymbol{x}). \quad (2.35)$$

Taking into account the recurrence relations (2.31) for the functions $v_0(\boldsymbol{x}), ..., v_K(\boldsymbol{x})$ we get (*cf.* (2.21))

$$-\sum_{l=0}^{K-2} \varepsilon^l h_l\Big(\nabla v^{(K)}(\boldsymbol{x}), \nabla\nabla v^{(K)}(\boldsymbol{x}), ..., \nabla^{l+2} v^{(K)}(\boldsymbol{x})\Big) = f(\boldsymbol{x}) + \varepsilon^{K-1} r_{K-1}(\boldsymbol{x}), \quad (2.36)$$

where $r_{K-1}(\boldsymbol{x})$ is a polynomial of $\nabla^l v_1, ..., \nabla^l v_K$ with $1 \le l \le K$ and of the derivatives $(\nabla_{z_1}, ..., \nabla_{z_{l+2}})^l h_l(\boldsymbol{z}_1, ..., \boldsymbol{z}_{j+2})|_{z_1=z_1(x), ..., z_{l+2}=z_{l+2}(x)}$ with $1 \le l \le K$ and some uniformly bounded vector functions $\boldsymbol{z}_1(\boldsymbol{x}), ..., \boldsymbol{z}_{j+2}(\boldsymbol{x})$. Hence, the absolute value of the remainder $r_{K-1}(\boldsymbol{x})$ is uniformly bounded by some constant $\tilde{C}_{K-1}$.

From (2.35) and (2.36) we achieve the following equality

$$-\text{div } \boldsymbol{j}\left(\boldsymbol{x}/\varepsilon, \nabla u^{(K)}(\boldsymbol{x})\right) = f(\boldsymbol{x}) + \varepsilon^{K-1}\Big(R_{K-1}(\boldsymbol{x}/\varepsilon, \boldsymbol{x}) + r_{K-1}(\boldsymbol{x})\Big)$$

$$= f(\boldsymbol{x}) + \varepsilon^{K-1} \theta_{K-1}(\boldsymbol{x}, \varepsilon),$$

where $\Big|\theta_{K-1}(\boldsymbol{x}, \varepsilon)\Big| \le C_{K-1} = \hat{C}_{K-1} + \tilde{C}_{K-1}$.

The inequality (2.34) now follows from (2.8).

2.4 Some further remarks and prospects

2.4.1 Infinite-order homogenised solution

We execute an idea introduced in the paper by Smyshlyaev and Cherednichenko [42] to cancel the effect of rapid oscillations in the asymptotics (2.10) by considering a family of "translated" problems of the form (2.2) with a parameter $\boldsymbol{\zeta} \in Q$:

$$-\mathrm{div}\, \boldsymbol{j}\Big(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u^{\varepsilon}(\boldsymbol{x})\Big) = f(\boldsymbol{x}), \quad \varepsilon > 0. \tag{2.37}$$

For any $\boldsymbol{\zeta} \in Q$ the problem (2.37) has a unique solution $u^{\zeta,\varepsilon}(\boldsymbol{x})$. Consider the averaging of this solution with respect to the parameter $\boldsymbol{\zeta}$:

$$\bar{u}^{\varepsilon}(\boldsymbol{x}) = \int\limits_{Q} u^{\zeta,\varepsilon}(\boldsymbol{x})\mathrm{d}\boldsymbol{\zeta}.$$

Then for any $K = 0, 1, 2, ...$ the following estimate holds with some constant $C^{(K)} > 0$

$$\int\limits_{\mathbf{T}} \left(\bar{u}^{\varepsilon}(\boldsymbol{x}) - \sum_{s=0}^{K} \varepsilon^{s} v_{s}(\boldsymbol{x})\right)^{2} \mathrm{d}\boldsymbol{x} \le C^{(K)}\varepsilon^{2K}. \tag{2.38}$$

The proof is completely analogous to that given in Chapter 1.

Therefore, $\bar{u}^{\varepsilon}(\boldsymbol{x})$ may be called the infinite-order homogenised solution and (2.38) implies that the series (2.11) is the asymptotics of $\bar{u}^{\varepsilon}$.

2.4.2 Higher-order homogenised variational problems

In the same fashion as in Chapter 1 we can consider a family of variational problems with a parameter $\boldsymbol{\zeta} \in Q$:

$$I^{\zeta}(\varepsilon, f) = \min_{u(x)} E_{\varepsilon}^{\zeta}(u, f) = \min_{u \in W_{0,per}^{1,p}} \int\limits_{\mathbf{T}} \Big(W\Big(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})\Big) - f(\boldsymbol{x})u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}.$$

Introducing the $\boldsymbol{\zeta}$-averaged energy functional

$$\bar{I}(\varepsilon, f) = \int\limits_{Q} I^{\zeta}(\varepsilon, f)\mathrm{d}\boldsymbol{\zeta} = \min_{u(x,\zeta)} \int\limits_{Q} E_{\varepsilon}^{\zeta}(u, f)\mathrm{d}\zeta$$

$$= \min_{u(x,\zeta)} \int\limits_{Q}\int\limits_{\mathbf{T}} \Big(W\Big(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})\Big) - f(\boldsymbol{x})u(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}$$

and selecting $K \geq 2$, we restrict the last minimisation to the set

$$U^{(K)} = \left\{ u(\boldsymbol{x}, \boldsymbol{\zeta}) \in W^{1,p}_{0,per}(\mathbf{T}) : u(\boldsymbol{x}, \boldsymbol{\zeta}) = v(\boldsymbol{x}) + \sum_{l=1}^{K-1} \varepsilon^l u_l\left(\frac{\boldsymbol{x}}{\varepsilon} + \boldsymbol{\zeta}, \nabla v(\boldsymbol{x}), ..., \nabla^l v(\boldsymbol{x})\right) \right\}, \tag{2.39}$$

where $v(\boldsymbol{x}) \in W^{K,p}_{0,per}(\mathbf{T})$, the set of all $\mathbf{T}-$periodic functions from the Sobolev space $W^{K,p}_{loc}(\mathbf{R}^d)$. Then

$$\min_{u(x,\zeta)\in U^{(K)}} \int_{\mathbf{T}} \int_{Q} \left(W\left(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla u(\boldsymbol{x})\right) - f(\boldsymbol{x})u(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\zeta}$$

$$= \min_{v(x)} \int_{\mathbf{T}} \left(\hat{W}^{(K)}_{\varepsilon}\left(\nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})\right) - f(\boldsymbol{x})v(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x}. \tag{2.40}$$

In the last formula

$$\hat{W}^{(K)}_{\varepsilon}\left(\nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})\right) = \left\langle W\left(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \Phi^{\varepsilon}_K\left(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})\right)\right) \right\rangle,$$

where $\Phi^{\varepsilon}_K\left(\boldsymbol{x}/\varepsilon + \boldsymbol{\zeta}, \nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})\right)$ is a finite sum in powers of ε akin to that in the definition of the set $U^{(K)}$.

The functional

$$\int_{\mathbf{T}} \left(\hat{W}^{(K)}_{\varepsilon}\left(\nabla v(\boldsymbol{x}), ..., \nabla^K v(\boldsymbol{x})\right) - f(\boldsymbol{x})v(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} \tag{2.41}$$

is convex in the linear case and the minimiser exists and is unique (see Chapter 1, Section 1.2.3). It is unknown if the minimisation problem (2.40) is well-posed in the nonlinear case. If so, then there exists a unique solution $v_K(\boldsymbol{x})$ to the problem (2.40) and it is natural to call it the homogenised solution of order K.

Otherwise, one may need to choose differently the truncation procedure leading to (2.39) and determining the minimisation set $U^{(K)}$ in (2.40).

2.4.3 Applications to non-uniformly elliptic problems

In our analysis so far we substantially used the ellipticity condition (2.4). However, in many applications potential functions $W(\boldsymbol{y}, \boldsymbol{e})$ arise that do not satisfy this condition at some points. One of the most well-known examples is the so called power-law potential

$$W(\boldsymbol{y}, \boldsymbol{e}) = \gamma(\boldsymbol{y})|\boldsymbol{e}|^p, \quad p > 1, \tag{2.42}$$

where $\gamma(\boldsymbol{y}) \geq \gamma_0 > 0$ is some smooth function. Potentials of this type are often used in high temperature creep or in deformation theory of plasticity with hardening in proportional loading. Although (as is shown in the Appendix F) the function (2.42) satisfies the condition (2.6), if $p > 2$ it does not satisfy the inequality (2.4) in the vicinity of the point $\boldsymbol{e} = \mathbf{0}$. (If $p < 2$ this condition fails for large $|\boldsymbol{e}|$.)

Nevertheless, we prove below that in the dimension two ($d = 2$) in the case of the potential (2.42) the expression $\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})$ (*cf.* (2.14)) does not take zero value *i.e.* $\left|\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\right| \geq \delta(\boldsymbol{z}) > 0$ for $\boldsymbol{z} \neq \mathbf{0}$. This implies that the construction of this chapter remains applicable provided $\nabla v_0(\boldsymbol{x}) \neq 0$, where $v_0(\boldsymbol{x})$ solves the homogenised equation (2.24).

Indeed, in this case the equation (2.14) takes the following form

$$\operatorname{div}_y\left(\gamma(\boldsymbol{y})\left|\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\right|^{p-2}\left(\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\right)\right) = 0. \tag{2.43}$$

By introducing the function $U(\boldsymbol{y}, \boldsymbol{z}) := \boldsymbol{z} \cdot \boldsymbol{y} + u_1(\boldsymbol{y}, \boldsymbol{z})$, we can rewrite the equation (2.43) as

$$\operatorname{div}_y\left(\gamma(\boldsymbol{y})\left|\nabla_y U(\boldsymbol{y}, \boldsymbol{z})\right|^{p-2}\nabla_y U(\boldsymbol{y}, \boldsymbol{z})\right) = 0. \tag{2.44}$$

It was shown by Alessandrini and Sigalotti [3] that a $W^{1,p}_{loc}(Q)$-solution $U(\boldsymbol{y})$ of the equation (2.44) belongs to the class $C^{1,\alpha}_{loc}(Q)$ for some $\alpha \in (0, 1]$. Therefore, $U(\boldsymbol{y}) \in C^{1,\alpha}_{per}(Q)$, where by $C^{1,\alpha}_{per}(Q)$ we denote the closure of the set of infinitely smooth Q-periodic functions in the space $C^{1,\alpha}(Q)$. This implies that the gradient $\nabla_y U$ of a periodic solution to the equation (2.44) can be understood in the classical sense, and the standard notion of a critical point of the function $U(\boldsymbol{y})$, *i.e.* such a point $\boldsymbol{y}$ at which $\nabla_y U = 0$, is well-defined. Another part of the result obtained by Alessandrini and Sigalotti was the fact that the critical points are isolated and have positive index (a variant of the maximum principle). By definition, the index of a regular domain D whose boundary does not contain any critical points is the integral

$$I(D, U) := \int_{\partial D} \mathrm{d}(\arg \nabla_y U),$$

The index of an isolated critical point $\boldsymbol{y}_0$ denoted by $I(\boldsymbol{y}_0, U)$ is the index of any regular domain containing the point $\boldsymbol{y}_0$ and no other critical points. Obviously, this definition does not depend on the choice of the domain D, containing the point $\boldsymbol{y}_0$. Also note that the index of a regular domain containing only isolated critical points is equal to the sum of indices of all the critical points it comprises.

We claim that in our situation, when $\boldsymbol{z} \neq \mathbf{0}$ the function $U(\boldsymbol{y})$ does not have critical points at all, which in other words means that $\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})$ does not vanish if $\boldsymbol{z} \neq \mathbf{0}$. To prove this we use a technique presented in the paper of Alessandrini and Nesi [2, Proposition 2]. Here we make an exposition of the proof adjusted to our particular situation.

Lemma.

Assume that $d = 2$ and let $u_1(\boldsymbol{y}, \boldsymbol{z})$ be a Q-periodic solution to the equation (2.43), where $\boldsymbol{z}$ is fixed and $\boldsymbol{z} \neq \mathbf{0}$. Then the function $U = \boldsymbol{z} \cdot \boldsymbol{y} + u_1(\boldsymbol{y}, \boldsymbol{z})$ does not have critical points in $\mathbf{R}^2$.

Proof:

Due to the fact that critical points of the function $U(\boldsymbol{y})$ are isolated, there is only a finite number of them in the square Q. Therefore, we can find a point $\bar{\boldsymbol{y}} = (\bar{y}_1, \bar{y}_2) \in Q$ such that the segments $\{\boldsymbol{y} \in \mathbf{R}^2 : y_1 = \bar{y}_1, |y_2| < 2\}$ and $\{\boldsymbol{y} \in \mathbf{R}^2 : y_2 = \bar{y}_2, |y_1| < 2\}$ do not contain any critical points. Then by periodicity of the vector $\nabla_y U$ we conclude that there are no critical points on the four translated segments $\{\boldsymbol{y} \in \mathbf{R}^2 : y_1 = \bar{y}_1 \pm 1, |y_2| < 2\}$ and $\{\boldsymbol{y} \in \mathbf{R}^2 : |y_1| < 2, y_2 = \bar{y}_2 \pm 1\}$. Therefore, the boundary of the domain

$$Q_1 := \{\boldsymbol{y} \in \mathbf{R}^2 : |y_1 - \bar{y}_1| < 1, |y_2 - \bar{y}_2| < 1\}$$

does not contain any critical points of the function $U(\boldsymbol{y})$.

Assuming that the function $U(\boldsymbol{y})$ has any critical points $\boldsymbol{y}^k, k = 1, ..., N$ in the square Q_1, then we get on one hand $I(Q_1, U) = \sum_{k=0}^{N} I(\boldsymbol{y}^k, U) > 0$ and on the other hand $I(Q_1, U) = \int\limits_{\partial D} \mathrm{d}(\arg \nabla_y U) = 0$ by periodicity of the vector $\nabla_y U$. Thus, our assumption is wrong and $U(\boldsymbol{y})$ does not have any critical points in Q_1, which by periodicity of the vector $\nabla_y U$ implies the conclusion that $U(\boldsymbol{y})$ does not have any critical points in the whole of $\mathbf{R}^2$. $\square$

If the function ∇v_0 does not take zero value, then the function v_0 is smooth and the functions $v_s, s = 1, ..., K$ are smooth, too. This follows from the fact that the matrix of second derivatives of the homogenised energy is non-negative and homogeneous of order $p - 2$. Therefore, it satisfies the following inequality with some positive constant c:

$$\frac{\partial^2 \hat{W}}{\partial z_i \partial z_j} \eta_i \eta_j \geq c |\boldsymbol{z}|^{p-2} \eta_i \eta_i \quad \text{for any } \boldsymbol{\eta} \in \mathbf{R}^d,$$

and due to the fact that $\nabla v_0 \neq \mathbf{0}$ for any $\boldsymbol{x} \in \mathbf{T}$, the equations (2.24), (2.30) and (2.31) for $v_s, s = 0, ..., K$ are uniformly elliptic and hence $v_s, s = 0, ..., K$ are smooth. Thus, a sufficient condition for the asymptotics (2.10)–(2.11) to hold in the two-dimensional power-law case is that the gradient of the solution to the variational problem (2.25) does not vanish.

Alternatively, if $d \neq 2$, the function $\boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})$ may vanish and hence the linear operator in the equations for the functions u_2, u_3, *etc* (see (2.20)) ceases to be uniformly elliptic. In this case methods of analysis in *weighted* spaces (see *e.g.* Zhikov [59]) may still be applicable to construct an asymptotic expansion of the solution to the problem (2.2) with the power-law potential (2.42). However, certain obstacles have been revealed in the attempt to carry out this idea, and we do not give the exposition of the related technique here.

Discussion

In this chapter we have shown that the techniques of remainder estimates of Bakhvalov and Panasenko for construction and rigorous justification of higher-order terms in the homogenised behaviour of periodic heterogeneous media can be carried over to the case of scalar non-linear equations in divergence form. These higher-order terms are given explicitly via a system of recurrence relations, which can be used for implementation in numerical algorithms.

The combination of variational and asymptotic approaches that was introduced in Chapter 1 can also be generalised to the nonlinear case to some extent. The natural "translation averaging" procedure proves to be useful again for constructing the infinite-order homogenised solution, which has a power-series asymptotics in ε analogous to the one considered in the linear case. This asymptotics has been supported by remainder estimates. The notion of a homogenised equation of higher order can be considered in the non-linear case as well, but turns out to be not as straightforward as in the linear case. In particular, an adequate way of making the variational truncation needs to be explored to complete this part of the work.

The case of a non-uniformly elliptic equation is important in applications and was given separate attention. In particular, the case of the power-law stored energy function proves to be tractable in the two-dimensional situation. Provided some easily verified condition is satisfied, the asymptotic and variational approaches discussed for the uniformly elliptic case are applicable to the power constitutive law as well. However, the case of higher dimensions turns out to be more delicate and needs further investigation in the future. On the other hand, it is known (see *e.g.* Zhikov [60]) that the loss of uniform ellipticity (in the linear problems) may lead to "non-classical" homogenised limits (involving *e.g. non-locality*). From this point of view, it is of interest to explore possible relations between nonlocal effects and higher-order terms in homogenised equations, as well as the possibility of non-classical effects in the *nonlinear* homogenisation. The next chapter is intended to be a step in this direction.

Appendix A: Proof of the inequality (2.28)

In this appendix we follow the argument of Bakhvalov and Panasenko [11].

Lemma.

Let a function $W = W(\boldsymbol{y}, \boldsymbol{e})$, $\boldsymbol{y}, \boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in y, $Q = [0,1]^d$ and satisfy the following inequality with a positive constant ν

$$\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j} \zeta_i \zeta_j \geq \nu \zeta_i \zeta_i \tag{2.45}$$

for any $\boldsymbol{y}, \boldsymbol{e} = (e_1, ..., e_d), \boldsymbol{\zeta} = (\zeta_1, ..., \zeta_d) \in \mathbf{R}^d$.

Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(y)} \left\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla\psi(\boldsymbol{y})\Big)\right\rangle \tag{2.46}$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{y})$.

Then the following inequality holds

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_i \partial z_j}\eta_i\eta_j \geq \nu\eta_i\eta_i \tag{2.47}$$

for any $\boldsymbol{z} = (z_1, ..., z_d), \boldsymbol{\eta} = (\eta_1, ..., \eta_d) \in \mathbf{R}^d$.

Proof:

Denote $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{y}, \boldsymbol{e})$. It is easy to see that the Euler-Lagrange equation for the minimisation problem (2.46) is

$$\mathrm{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y\psi(\boldsymbol{y}, \boldsymbol{z})\Big) = 0. \tag{2.48}$$

It obviously coincides with the equation (2.14), *i.e.* $\psi = u_1(\boldsymbol{y}, \boldsymbol{z})$ is the minimiser for the problem (2.46).

Let us introduce the following notation

$$\boldsymbol{Y}(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})$$

and substitute $\boldsymbol{e} = \boldsymbol{Y}(\boldsymbol{y}, \boldsymbol{z})$ and $\boldsymbol{\zeta} = \boldsymbol{Y}_{,z_q}\eta_q$ into the inequality (2.45).

Thus

$$\left.\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j}\right|_{e=Y(y,z)} \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_q}\eta_q \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r}\eta_r \geq \nu \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_q}\eta_q \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r}\eta_r.$$

Taking the average with respect to $\boldsymbol{y}$ over Q in the last inequality we get

$$\beta_{qr}(\boldsymbol{z})\eta_q\eta_r \geq \nu \left\langle \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_q}\eta_q \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r}\eta_r \right\rangle, \tag{2.49}$$

where

$$\beta_{qr}(\boldsymbol{z}) = \left\langle \left.\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j}\right|_{e=Y(y,z)} \frac{\partial Y_i(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_q} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \right\rangle$$

$$= \left\langle \left.\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j}\right|_{e=Y(y,z)} \left(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_i \partial z_q}\right) \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \right\rangle$$

$$= \left\langle \left.\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_q \partial e_j}\right|_{e=Y(y,z)} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \right\rangle + \left\langle \left.\frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j}\right|_{e=Y(y,z)} \frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_i \partial z_q} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial z_r} \right\rangle \tag{2.50}$$

We claim that the second term in (2.50) is identically zero. To verify this consider the equation (2.48). Multiply its both sides by some arbitrary function $\phi = \phi(\boldsymbol{y})$, take the average with respect to $\boldsymbol{y}$ over Q, and integrate by parts. We get

$$\left\langle \boldsymbol{j}\left(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\right) \cdot \nabla\phi(\boldsymbol{y}) \right\rangle = 0.$$

Differentiate the last equality with respect to z_r and note that $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e}) = \nabla_e W(\boldsymbol{y}, \boldsymbol{e})$. We come to the following equality

$$\left\langle \left. \frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j} \right|_{e=Y(y,z)} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \frac{\partial \phi(\boldsymbol{y})}{\partial y_i} \right\rangle = 0. \tag{2.51}$$

This equality holds for all $\boldsymbol{z} \in \mathbf{R}^d$. Now, set $\phi_q(\boldsymbol{y}, \boldsymbol{z}) = \left(u_1(\boldsymbol{y}, \boldsymbol{z})\right)_{,z_q}$ for every $\boldsymbol{z} \in \mathbf{R}^d$, $q = 1, ..., d$. Substituting the functions $\phi_q(\boldsymbol{y}, \boldsymbol{z})$ instead of $\phi(\boldsymbol{y})$ into the identity (2.51) we successively get

$$\left\langle \left. \frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_i \partial e_j} \right|_{e=Y(y,z)} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_i \partial z_q} \right\rangle = 0$$

for all $\boldsymbol{z} \in \mathbf{R}^d, q = 1, ..., d$. Hence,

$$\beta_{qr}(\boldsymbol{z}) = \left\langle \left. \frac{\partial^2 W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_q \partial e_j} \right|_{e=Y(y,z)} \frac{\partial Y_j(\boldsymbol{y}, \boldsymbol{z})}{\partial \boldsymbol{z}_r} \right\rangle.$$

To verify that the following identity holds

$$\beta_{qr}(\boldsymbol{z}) = \frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r},$$

differentiate the equality

$$\hat{W}(\boldsymbol{z}) = \left\langle W\left(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\right) \right\rangle$$

to get

$$\frac{\partial \hat{W}(\boldsymbol{z})}{\partial z_q} = \left\langle \left. \frac{\partial W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_j} \right|_{e=Y(y,z)} \left(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_i \partial z_q} \right) \right\rangle$$

$$= \left\langle \left. \frac{\partial W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_q} \right|_{e=Y(y,z)} \right\rangle + \left\langle \left. \frac{\partial W(\boldsymbol{y}, \boldsymbol{e})}{\partial e_j} \right|_{e=Y(y,z)} \frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_i \partial z_q} \right\rangle.$$

Integrate by parts in the second term of the last sum and note that

$$\mathrm{div}_y \left(\nabla_e W(\boldsymbol{y}, \boldsymbol{e})|_{e=Y(y,z)} \right) = 0$$

in view of (2.48). Thus,

$$\left\langle \left.\frac{\partial W(\boldsymbol{y},\boldsymbol{e})}{\partial e_j}\right|_{e=Y(y,z)} \frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_i \partial z_q} \right\rangle = 0$$

for all $\boldsymbol{z} \in \mathbf{R}^d$ and so

$$\frac{\partial \hat{W}(\boldsymbol{z})}{\partial z_q} = \left\langle \left.\frac{\partial W(\boldsymbol{y},\boldsymbol{e})}{\partial e_q}\right|_{e=Y(y,z)} \right\rangle.$$

Differentiating the last equality one more time we obtain the following identity

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r} = \left\langle \left.\frac{\partial^2 W(\boldsymbol{y},\boldsymbol{e})}{\partial e_q \partial e_j}\right|_{e=Y(y,z)} \frac{\partial Y_j(\boldsymbol{y},\boldsymbol{z})}{\partial z_r} \right\rangle,$$

which immediately implies

$$\beta_{qr}(\boldsymbol{z}) = \frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r}$$

for all $\boldsymbol{z} \in \mathbf{R}^d$.

Note finally that the following estimate holds

$$\left\langle \left(\frac{\partial Y_i}{\partial z_q}\eta_q\right)^2 \right\rangle = \left\langle \left(\delta_{iq} + \frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_i \partial z_q}\right)^2 \eta_q^2 \right\rangle = \left\langle \eta_i^2 + 2\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_i \partial z_i} + \left(\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_i \partial z_q}\right)^2 \eta_q^2 \right\rangle$$

$$= \eta_i^2 + \left\langle \left(\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_i \partial z_q}\right)^2 \eta_q^2 \right\rangle \geq \eta_i^2$$

for any $i = 1, ..., d$.

Now, taking into account (2.49) we get

$$\frac{\partial^2 \hat{W}(\boldsymbol{z})}{\partial z_q \partial z_r}\eta_q \eta_r \geq \nu \eta_i \eta_i$$

as required. □

Appendix B: Growth condition for the homogenised energy

In this section we show that the homogenised energy $\hat{W}(\boldsymbol{z})$ satisfies a standard growth condition (2.26) of the same form as for the original heterogeneous stored energy function $W(\boldsymbol{y},\boldsymbol{e})$ (see (2.3)).

Lemma.

Let a function $W = W(\boldsymbol{y},\boldsymbol{e})$, $\boldsymbol{y},\boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in $\boldsymbol{y}$, $Q = [0,1]^d$. Suppose it satisfies the following estimates

$$-A_1 + B_1|\boldsymbol{e}|^p \leq W(\boldsymbol{y},\boldsymbol{e}) \leq A_2 + B_2|\boldsymbol{e}|^p \quad \text{for any } \boldsymbol{y},\boldsymbol{e} \in \mathbf{R}^d \tag{2.52}$$

with some positive constants A_1, A_2, B_1, B_2 and $p \geq 1$.

Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(y)} \Big\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla\psi(\boldsymbol{y})\Big)\Big\rangle$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{y})$. Then

$$-A_1 + B_1|\boldsymbol{z}|^p \leq \hat{W}(\boldsymbol{z}) \leq A_2 + B_2|\boldsymbol{z}|^p \text{ for any } \boldsymbol{z} \in \mathbf{R}^d. \tag{2.53}$$

Proof:

Taking $\psi(\boldsymbol{y}) \equiv 0$ we conclude that

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(y)} \Big\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla\psi(\boldsymbol{y})\Big)\Big\rangle \leq \Big\langle W(\boldsymbol{y}, \boldsymbol{z})\Big\rangle \leq A_2 + B_2|\boldsymbol{z}|^p.$$

so the right-hand inequality in (2.53) is proved.

To prove the left-hand part of (2.53) we remark that the following inequalities hold

$$\Big\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla\psi(\boldsymbol{y})\Big)\Big\rangle \geq \Big\langle -A_1 + B_1|\boldsymbol{z} + \nabla\psi|^p\Big\rangle \geq -A_1 + B_1\Big\langle|\boldsymbol{z} + \nabla\psi|\Big\rangle^p$$

$$\geq -A_1 + B_1\Big|\langle\boldsymbol{z} + \nabla\psi\rangle\Big|^p \geq -A_1 + B_1|\boldsymbol{z}|^p. \tag{2.54}$$

Here, we first used the given lower bound for $W(\boldsymbol{y}, \boldsymbol{e})$, then we applied the Hölder inequality, used a basic property of the mean value, and finally took advantage of the fact that the function $\psi(\boldsymbol{y})$ is Q-periodic and therefore the mean of its gradient is zero.

Taking the infima with respect to $\psi(\boldsymbol{y})$ of the first and the last expressions in (2.54) we get the required lower bound. □

Appendix C: Strong monotonicity of the homogenised energy

We aim here at proving the following lemma.

Lemma.

Let a function $W = W(\boldsymbol{y}, \boldsymbol{e})$, $\boldsymbol{y}, \boldsymbol{e} \in \mathbf{R}^d$ be Q-periodic in $\boldsymbol{y}$, $Q = [0,1]^d$ and strongly monotonic, *i.e.*

$$\Big(\nabla_e W(\boldsymbol{y}, \boldsymbol{e}_1) - \nabla_e W(\boldsymbol{y}, \boldsymbol{e}_2)\Big) \cdot \Big(\boldsymbol{e}_1 - \boldsymbol{e}_2\Big) \geq \alpha|\boldsymbol{e}_1 - \boldsymbol{e}_2|^p, \quad \alpha > 0, \quad p > 1 \tag{2.55}$$

for every $\boldsymbol{y} \in \mathbf{T}$ and any $\boldsymbol{e}_1, \boldsymbol{e}_1 \in \mathbf{R}^d$. Define the function $\hat{W}(\boldsymbol{z})$ as follows

$$\hat{W}(\boldsymbol{z}) = \inf_{\psi(y)} \Big\langle W\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla\psi(\boldsymbol{y})\Big)\Big\rangle \tag{2.56}$$

where the infimum is taken over the set of all Q-periodic functions $\psi(\boldsymbol{y})$.

Then the function $\hat{W}(\boldsymbol{z})$ is also strongly monotonic:

$$\left(\nabla\hat{W}(\boldsymbol{z}_1)-\nabla\hat{W}(\boldsymbol{z}_2)\right)\cdot\left(\boldsymbol{z}_1-\boldsymbol{z}_2\right)\geq\alpha|\boldsymbol{z}_1-\boldsymbol{z}_2|^p. \tag{2.57}$$

Note that the parameters α and p in (2.57) are the same as in (2.55).

Proof:

Apply averaging with respect to $\boldsymbol{y}\in Q$ to the inequality (2.55) where $\boldsymbol{e}_1=\boldsymbol{z}_1+u_1(\boldsymbol{y},\boldsymbol{z}_1)$ and $\boldsymbol{e}_2=\boldsymbol{z}_2+u_1(\boldsymbol{y},\boldsymbol{z}_2)$.

Using the identity

$$\nabla\hat{W}(\boldsymbol{z})=\left\langle\nabla_e W|_{e=z+u_1(y,z)}\right\rangle$$

proved in the Appendix A and the fact that the function u_1 is a solution of the equation (2.48), we arrive at the following inequality

$$\left(\nabla\hat{W}(\boldsymbol{z}_1)-\nabla\hat{W}(\boldsymbol{z}_2)\right)\cdot\left(\boldsymbol{z}_1-\boldsymbol{z}_2\right)=\Bigg\langle\Big(\nabla_e W(\boldsymbol{y},\boldsymbol{e})|_{e=z_1+\nabla_y u_1(y,z_1)}$$

$$-\nabla_e W(\boldsymbol{y},\boldsymbol{e})|_{e=z_1+u_1(y,z_1)}\Big)\cdot\Big(\boldsymbol{z}_1+\nabla_y u_1(\boldsymbol{y},\boldsymbol{z}_1)-\boldsymbol{z}_2-\nabla_y u_1(\boldsymbol{y},\boldsymbol{z}_2)\Big)\Bigg\rangle$$

$$\geq\alpha\Big\langle\Big|\boldsymbol{z}_1-\boldsymbol{z}_2+\nabla_y u_1(\boldsymbol{y},\boldsymbol{z}_1)-\nabla_y u_1(\boldsymbol{y},\boldsymbol{z}_2)\Big|^p\Big\rangle.$$

Integrating by parts in the second term of the left-hand side of the last inequality and using the fact that

$$\operatorname{div}_y\Big(\nabla_e W(\boldsymbol{y},\boldsymbol{e})|_{e=z+\nabla_y u_1(y,z)}\Big)=0\quad\text{for any }\boldsymbol{z}\in\mathbf{R}^d$$

we conclude that this term is zero.

Finally, from the periodicity of the functions $u_1(\boldsymbol{y},\boldsymbol{z}_1)$ and $u_1(\boldsymbol{y},\boldsymbol{z}_2)$ with respect to $\boldsymbol{y}$ we get (using Hölder inequality)

$$\Big\langle\Big|\boldsymbol{z}_1-\boldsymbol{z}_2+\nabla u_1(\boldsymbol{y},\boldsymbol{z}_1)-\nabla u_1(\boldsymbol{y},\boldsymbol{z}_2)\Big|^p\Big\rangle$$

$$\geq\Big|\Big\langle\boldsymbol{z}_1-\boldsymbol{z}_2+\nabla u_1(\boldsymbol{y},\boldsymbol{z}_1)-\nabla u_1(\boldsymbol{y},\boldsymbol{z}_2)\Big\rangle\Big|^p=|\boldsymbol{z}_1-\boldsymbol{z}_2|^p,$$

that gives us the required right-hand side in (2.57). □

Appendix D: A bound for the gradient of a convex function having power growth

The result of this appendix is contained in the following lemma.

Lemma.

Let $W = W(e)$ be a smooth convex function satisfying the following inequality

$$-A_1 + B_1|e|^p \leq W(e) \leq A_2 + B_2|e|^p \quad \text{for any } e \in \mathbf{R}^d$$

with some positive constants A_1, A_2, B_1, B_2 and $p > 1$.

Then there exist positive constants A and B, which are dependent only on the values of A_1, A_2, B_1, B_2 and p, such that

$$\Big|\nabla W(e)\Big| \leq A + B|e|^{p-1} \quad \text{for any} \quad e \in \mathbf{R}^d. \tag{2.58}$$

Proof:

Let us first introduce the following notation $e = (e_1, e')$, $W_1(e) := -A_1 + B_1|e|^p$, $W_2(e) := A_2 + B_2|e|^p, f_1(e_1) := W_1(e_1, e'), f_2(e_1) := W_2(e_1, e'), f(e_1) := W(e_1, e')$, $a := \min\Big\{0, \min\{e_1 : f'(e_1) = 0\}\Big\}$, and $b := \max\Big\{0, \max\{e_1 : f'(e_1) = 0\}\Big\}$.

Let us assume first that $\hat{e}_1 > b$. Obviously,

$$\frac{\partial W(e)}{\partial e_1}\bigg|_{e=(\hat{e}_1, e')} = f'(\hat{e}_1) \leq f_2'(\bar{e}_1) = \frac{\partial W_2(e)}{\partial e_1}\bigg|_{e=(\bar{e}_1, e')}, \tag{2.59}$$

where $\Big(\bar{e}_1, f_2(\bar{e}_1)\Big)$ is the point at which the tangent line to the graph of the function $f_2(e_1)$ passing through the point $\Big(\hat{e}_1, f_1(\hat{e}_1)\Big)$ grazes the graph of the function $f_2(e_1)$ (see Figure (2.1)).

It is not difficult to see that the values of $\hat{e}_1$ and $\bar{e}_1$ are linked by the following nonlinear equation

$$A_2 + B_2|\bar{e}|^p + A_1 - B_1|\hat{e}|^p = B_2 p|\bar{e}|^{p-2}\bar{e}_1(\bar{e}_1 - \hat{e}_1), \tag{2.60}$$

where $\bar{e} := (\bar{e}_1, e')$ and $\hat{e} := (\hat{e}_1, e')$.

Notice that if

$$|\bar{e}| \geq \left(\frac{A_1 + A_2}{B_2}\frac{2}{p-1}\right)^{\frac{1}{p}}$$

then

$$A_1 + A_2 + B_2|\bar{e}|^p - B_1|\hat{e}|^p \leq A_1 + A_2 + B_2|\bar{e}|^p \leq \frac{p+1}{2}B_2|\bar{e}|^p$$

and therefore from (2.60) we get

$$B_2 p|\bar{e}|^{p-2}\bar{e}_1(\bar{e}_1 - \hat{e}_1) \leq \frac{p+1}{2}B_2|\bar{e}|^p.$$

The last inequality can be rewritten as

$$p\bar{e}_1(\bar{e}_1 - \hat{e}_1) \leq \frac{p+1}{2}|\bar{e}|^2. \tag{2.61}$$

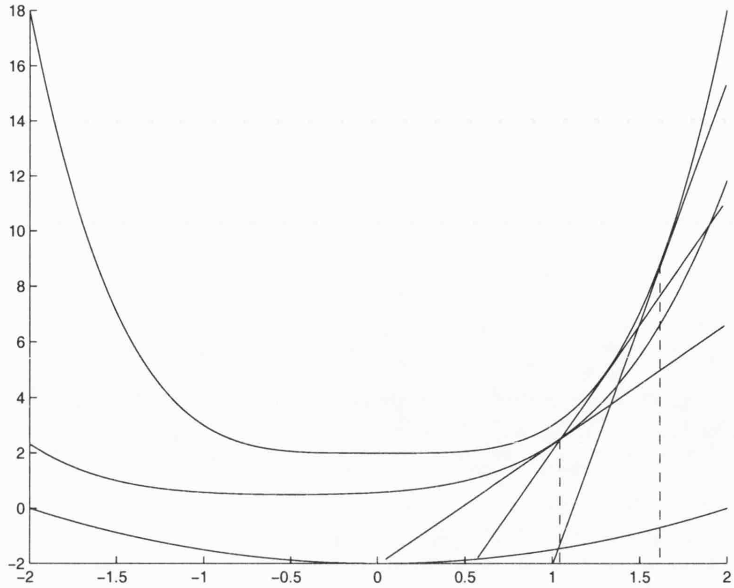

Figure 2.1: The graph of the function $f(e_1)$ is situated between the graphs of the functions $f_1(e_1)$ and $f_2(e_1)$. Therefore, the slope of the tangent to the graph of the function $f(e_1)$ at the point $(\hat{e}_1, f(\hat{e}_1))$ is smaller than the slope of the tangent to the graph of the function $f_2(e_1)$ passing through the point $(\hat{e}_1, f(\hat{e}_1))$, which is in turn smaller than the slope of the tangent to the graph of the function $f_2(e_1)$ passing through the point $(\hat{e}_1, f_1(\hat{e}_1))$.

If $\bar{e}_1 > M(p)|\boldsymbol{e}'|$, where $M(p) := \frac{1}{2}\Big(\sqrt{\frac{3p+1}{2(p+1)}} - 1\Big)^{-1}$, then

$$\sqrt{\frac{p+1}{2}}|\bar{\boldsymbol{e}}| = \sqrt{\frac{p+1}{2}}\Big(\bar{e}_1^2 + |\boldsymbol{e}'|^2\Big)^{\frac{1}{2}} \leq \sqrt{\frac{p+1}{2}}\bar{e}_1\left(1 + \frac{1}{2}\frac{|\boldsymbol{e}'|^2}{\bar{e}_1^2}\right) \leq \frac{\sqrt{3p+1}}{2}\bar{e}_1$$

and from the inequality (2.61) we get

$$p\bar{e}_1\hat{e}_1 \geq p\bar{e}_1^2 - \frac{p+1}{2}|\bar{\boldsymbol{e}}|^2 \geq p\bar{e}_1^2 - \frac{3p+1}{4}\bar{e}_1^2 = \frac{p-1}{4}\bar{e}_1^2,$$

from where

$$\bar{e}_1 \leq \frac{4p}{p-1}\hat{e}_1,$$

and consequently,

$$\frac{\partial W(\boldsymbol{y},\boldsymbol{e})}{\partial e_1}\bigg|_{e=(\hat{e}_1,e')} \leq \frac{\partial W_2(\boldsymbol{e})}{\partial e_1}\bigg|_{e=(\bar{e}_1,e')} \leq B_2 p|\bar{\boldsymbol{e}}|^{p-2}\bar{e}_1 \leq B_2 p\Big(\frac{4p}{p-1}\Big)^{p-1}|\hat{\boldsymbol{e}}|^{p-1}$$

Alternatively, if $\bar{e}_1 \leq M(p)|\boldsymbol{e}'|$ then

$$\frac{\partial W(\boldsymbol{y},\boldsymbol{e})}{\partial e_1}\bigg|_{e=(\hat{e}_1,e')} \leq \frac{\partial W_2(\boldsymbol{e})}{\partial e_1}\bigg|_{e=(\bar{e}_1,e')} \leq B_2 p|\bar{\boldsymbol{e}}|^{p-2}\bar{e}_1 = B_2 p\Big(\bar{e}_1^2 + |\boldsymbol{e}'|^2\Big)^{\frac{p-2}{2}}\bar{e}_1$$

$$\leq B_2 p\Big(\big(M(p)\big)^2+1\Big)^{\frac{p-2}{2}} M(p)|e'|^{p-1} \leq B_2 p\Big(\big(M(p)\big)^2+1\Big)^{\frac{p-2}{2}} M(p)|\hat{e}|^{p-1}.$$

Now, setting

$$\tilde{A} := B_2 p\Big(\frac{A_1+A_2}{B_2}\frac{2}{p-1}\Big)^{\frac{p-1}{p}}$$

and

$$\tilde{B} := \max\Bigg(B_2 p\Big(\frac{4p}{p-1}\Big)^{p-1}, B_2 p\Big(\big(M(p)\big)^2+1\Big)^{\frac{p-2}{2}} M(p)\Bigg)$$

we get the estimate

$$\frac{\partial W(e)}{\partial e_1} \leq \tilde{A} + \tilde{B}|e|^{p-1} \quad \text{for} \quad e_1 > b.$$

Note that using simple manipulations it is easy to verify that

$$\Big(\frac{4p}{p-1}\Big)^{p-1} > \Big(\big(M(p)\big)^2+1\Big)^{\frac{p-2}{2}} M(p) \quad \text{for} \quad p > 1,$$

and therefore

$$\tilde{B} = B_2 p\Big(\frac{4p}{p-1}\Big)^{p-1}.$$

The case $e_1 < a$ is considered in the way completely analogous to the one presented above, and we get

$$\frac{\partial W(e)}{\partial e_1} \geq -\tilde{A} - \tilde{B}|e|^{p-1} \quad \text{for} \quad e_1 < a.$$

Finally, if $a \leq e_1 \leq b$ then

$$\left|\frac{\partial W(e)}{\partial e_1}\right| \leq \left|\frac{\partial W(e)}{\partial e_1}\Big|_{e=(0,e')}\right| \leq \tilde{A}.$$

The fact that we considered the coordinate e_1 did not have any importance in the previous considerations. Therefore, for any $i = 1, ..., d$ we get

$$\left|\frac{\partial W(e)}{\partial e_i}\right| \leq \tilde{A} + \tilde{B}|e|^{p-1} \text{ for any } \quad e \in \mathbf{R}^d.$$

Hence, setting $A = \tilde{A}\sqrt{d}, B = \tilde{B}\sqrt{d}$ we obtain the required estimate (2.58). □

Appendix E: The function $u_1(y, z)$ is smooth with respect to the *pair* of arguments y and z

To prove that the function $u_1(y, z)$ is smooth with respect to the variable $z \in \mathbf{R}^d$ we are going to implement the classical implicit function theorem (J. Sivaloganathan, personal communication). To this end, we consider the following operator

$$G(u,\boldsymbol{z}) := \operatorname{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u(\boldsymbol{y})\Big)$$

acting from the product $C^{(2)}_{per,0}(Q) \times \mathbf{R}^d$ into the space $C_{per,0}(Q)$. Here the functional spaces on Q with the subscripts $per, 0$ are the closures of the set of infinitely differentiable Q-periodic functions in $\mathbf{R}^d$ having zero mean over Q with respect to the norm in the corresponding space without the subscript.

As long as we are going to resolve the equation $G(u,\boldsymbol{z}) = 0$ with respect to $\boldsymbol{z}$, we first should make sure that the operator G takes zero value at any point of the space $C^{(2)}_{per,0}(Q) \times \mathbf{R}^d$ in the neighbourhood of which we would like to justify the implicit function given by the equation $G(u,\boldsymbol{z}) = 0$. This is an easy task since the function $u_1(\boldsymbol{y},\boldsymbol{z})$ satisfies the above equation for any $\boldsymbol{z} \in \mathbf{R}^d$. Note that as we already noticed before (see *e.g.* Ladyzhenskaya & Uraltseva [30]), the function $u_1(\boldsymbol{y},\boldsymbol{z})$ is smooth with respect to $\boldsymbol{y}$ and therefore belongs to the required functional space.

Secondly, we should find the first partial differential of this operator with respect to the argument u and show that it is invertible at the point $(u_1, \boldsymbol{z})$. For this, we consider the increment of the functional G under an increment $h \in C^{(2)}_{per,0}(Q)$ of its argument u at the point $u = u_1$

$$
\begin{aligned}
G(u_1+h,\boldsymbol{z}) &= \operatorname{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y},\boldsymbol{z}) + \nabla_y h(\boldsymbol{y})\Big) = \operatorname{div}_y \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})\big|_{e=z+\nabla_y u_1(y,z)+\nabla_y h(y)} \\
&\quad + \frac{\partial j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s}\bigg|_{e=z+\nabla_y u_1(y,z)+\nabla_y h(y)} \left(\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_s \partial y_r} + \frac{\partial^2 h(\boldsymbol{y})}{\partial y_s \partial y_r}\right) \\
&= \operatorname{div}_y \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})\big|_{e=z+\nabla_y u_1(y,z)} + \frac{\partial}{\partial e_s}\operatorname{div}_y \boldsymbol{j}\,(\boldsymbol{y},\boldsymbol{e})\bigg|_{e=z+\nabla_y u_1(y,z)} \frac{\partial h(\boldsymbol{y})}{\partial y_s} \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial e_s \partial e_m}\operatorname{div}_y \boldsymbol{j}\,(\boldsymbol{y},\boldsymbol{e})\bigg|_{e=z+\nabla_y u_1(y,z)+\theta_1 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_s}\frac{\partial h(\boldsymbol{y})}{\partial y_m} \\
&+ \frac{\partial j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s}\bigg|_{e=z+\nabla_y u_1(y,z)} \frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_s \partial y_r} + \frac{\partial^2 j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s \partial e_m}\bigg|_{e=z+\nabla_y u_1(y,z)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_s \partial y_r} \\
&\quad + \frac{1}{2}\frac{\partial^3 j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s \partial e_m \partial e_l}\bigg|_{e=z+\nabla_y u_1(y,z)+\theta_2 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial h(\boldsymbol{y})}{\partial y_l}\frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_s \partial y_r} \\
&+ \frac{\partial j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s}\bigg|_{e=z+\nabla_y u_1(y,z)} \frac{\partial^2 h(\boldsymbol{y})}{\partial y_s \partial y_r} + \frac{\partial^2 j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s \partial e_m}\bigg|_{e=z+\nabla_y u_1(y,z)+\theta_3 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial^2 h(\boldsymbol{y})}{\partial y_s \partial y_r},
\end{aligned}
\tag{2.62}
$$

where $\theta_1 = \theta_1(u_1, h) \in [0,1]$, $\theta_2 = \theta_2(u_1, h) \in [0,1]$ and $\theta_3 = \theta_3(u_1, h) \in [0,1]$. Here we used the classical Lagrange's formula for the remainder term of Taylor expansion. The terms of zeroth order with respect to h in the last sum are zero altogether as follows

$$\operatorname{div}_y \boldsymbol{j}(\boldsymbol{y},\boldsymbol{e})\big|_{e=z+\nabla_y u_1(y,z)} + \frac{\partial j_r(\boldsymbol{y},\boldsymbol{e})}{\partial e_s}\bigg|_{e=z+\nabla_y u_1(y,z)} \frac{\partial^2 u_1(\boldsymbol{y},\boldsymbol{z})}{\partial y_s \partial y_r}$$

$$= \mathrm{div}_y \boldsymbol{j}\Big(\boldsymbol{y}, \boldsymbol{z} + \nabla_y u_1(\boldsymbol{y}, \boldsymbol{z})\Big) = 0.$$

The nonlinear part of the remaining sum in the right-hand side of (2.62) is

$$N(u_1, h) := \frac{1}{2}\frac{\partial^2}{\partial e_s \partial e_m}\mathrm{div}_y \boldsymbol{j}\,(\boldsymbol{y}, \boldsymbol{e})\,\Big|_{e=z+\nabla_y u_1(y,z)+\theta_1 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_s}\frac{\partial h(\boldsymbol{y})}{\partial y_m} +$$

$$+\frac{1}{2}\frac{\partial^3 j_r(\boldsymbol{y}, \boldsymbol{e})}{\partial e_s \partial e_m \partial e_l}\Big|_{e=z+\nabla_y u_1(y,z)+\theta_2 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial h(\boldsymbol{y})}{\partial y_l}\frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_s \partial y_r}$$

$$+\frac{\partial^2 j_r(\boldsymbol{y}, \boldsymbol{e})}{\partial e_s \partial e_m}\Big|_{e=z+\nabla_y u_1(y,z)+\theta_3 \nabla_y h(y)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial^2 h(\boldsymbol{y})}{\partial y_s \partial y_r}.$$

It is not difficult to see that due to smoothness of the vector function $\boldsymbol{j}(\boldsymbol{y}, \boldsymbol{e})$

$$\Big|N(u_1, h)\Big| \le C\|h\|^2_{C^{(2)}_{per}(Q)}$$

and therefore,the linear part of the sum in the right-hand side of (2.62) is the value of the partial Fréchet's differential of the operator $G(u, \boldsymbol{z})$ with respect to u at the point $u = u_1$ with the increment h

$$d_u G(u, \boldsymbol{z})\Big|_{u=u_1}[h] = \frac{\partial}{\partial e_s}\mathrm{div}_y \boldsymbol{j}\,(\boldsymbol{y}, \boldsymbol{e})\,\Big|_{e=\nabla_y u_1} \frac{\partial h(\boldsymbol{y})}{\partial y_s}$$

$$+\frac{\partial^2 j_r(\boldsymbol{y}, \boldsymbol{e})}{\partial e_s \partial e_m}\Big|_{e=\nabla_y u_1(y,z)} \frac{\partial h(\boldsymbol{y})}{\partial y_m}\frac{\partial^2 u_1(\boldsymbol{y}, \boldsymbol{z})}{\partial y_s \partial y_r}$$

$$+\frac{\partial j_r(\boldsymbol{y}, \boldsymbol{e})}{\partial e_s}\Big|_{e=\nabla_y u_1(y,z)} \frac{\partial^2 h(\boldsymbol{y})}{\partial y_s \partial y_r} = \frac{\partial}{\partial y_r}\left(\frac{\partial j_r(\boldsymbol{y}, \boldsymbol{e})}{\partial e_m}\frac{\partial h(\boldsymbol{y})}{\partial y_m}\right).$$

The last expression belongs to the space $C_{per,0}(Q)$ and it generates a positive definite quadratic form on $C^{(2)}_{per,0}(Q)$ by means of multiplying it by h and integrating by parts.

Thus, we have shown that the implicit function theorem is applicable and using the fact that the operator $G(u, \boldsymbol{z})$ is smooth with respect to $\boldsymbol{z}$ we conclude that the implicit operator $\boldsymbol{z} \longmapsto u_1(\boldsymbol{y}, \boldsymbol{z})$ is smooth from $\mathbf{R}^d$ into $C^{(2)}_{per,0}(Q)$. In particular, $u_1(y, \boldsymbol{z})$ is a smooth function of $\boldsymbol{z}$.

Appendix F: Monotonicity of the gradient of the power-law energy function

This appendix is devoted to the proof of the fact that for the power law (2.42) the inequality (2.6) is satisfied.

Lemma.

For any $p \in [2, +\infty)$ there exists $\alpha(p) > 0$ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^d$

$$\Big(|\boldsymbol{a}|^{p-2}\boldsymbol{a} - |\boldsymbol{b}|^{p-2}\boldsymbol{b}\Big) \cdot \Big(\boldsymbol{a} - \boldsymbol{b}\Big) \ge \alpha(p)|\boldsymbol{a} - \boldsymbol{b}|^p. \tag{2.63}$$

Proof:

Without loss of generality we can assume that $\boldsymbol{b} \neq \mathbf{0}$. Dividing both parts of (2.63) by $|\boldsymbol{b}|^p$ we get

$$\left(\frac{|\boldsymbol{a}|^{p-2}}{|\boldsymbol{b}|^{p-2}}\frac{\boldsymbol{a}}{|\boldsymbol{b}|} - \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{a}}{|\boldsymbol{b}|} - \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) \geq \alpha(p)\left|\frac{\boldsymbol{a}}{|\boldsymbol{b}|} - \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right|^p.$$

Let us denote $\frac{\boldsymbol{a}}{|\boldsymbol{b}|} =: \boldsymbol{c}$, $\frac{\boldsymbol{b}}{|\boldsymbol{b}|} =: \boldsymbol{e}(\boldsymbol{b})$, where $\boldsymbol{e}(\boldsymbol{b})$ reads "the unit vector along $\boldsymbol{b}$". Thus, the proposition can be formulated as follows. For any $\boldsymbol{c}, \boldsymbol{b} \in \mathbf{R}^d$

$$\Big(|\boldsymbol{c}|^{p-2}\boldsymbol{c} - \boldsymbol{e}(\boldsymbol{b})\Big) \cdot \Big(\boldsymbol{c} - \boldsymbol{e}(\boldsymbol{b})\Big) \geq \alpha(p)\Big|\boldsymbol{c} - \boldsymbol{e}(\boldsymbol{b})\Big|^p. \tag{2.64}$$

There exists an orthogonal matrix O such that $\boldsymbol{e}(\boldsymbol{b}) = O\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the unit vector of the first coordinate axis. Let us denote $\tilde{\boldsymbol{c}} := O^\top \boldsymbol{c}$. Due to the fact that the matrix O is a bijection between $\mathbf{R}^d$ and itself and also in virtue of the invariance of the inequality (2.64) with respect to rotations we can finally reformulate our problem in the following way.

Prove that there exists $\alpha(p) > 0$ such that for any vector $\tilde{\boldsymbol{c}} \in \mathbf{R}^d$

$$\Big(|\tilde{\boldsymbol{c}}|^{p-2}\tilde{\boldsymbol{c}} - \boldsymbol{e}_1\Big) \cdot \Big(\tilde{\boldsymbol{c}} - \boldsymbol{e}_1\Big) \geq \alpha(p)|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p.$$

We claim that, equivalently,

$$\inf_{\tilde{\boldsymbol{c}} \in \mathbf{R}^d \setminus \{\boldsymbol{e}_1\}} A(\tilde{\boldsymbol{c}}, p) = \alpha(p) > 0,$$

where

$$A(\tilde{\boldsymbol{c}}, p) := \frac{(|\tilde{\boldsymbol{c}}|^{p-2}\tilde{\boldsymbol{c}} - \boldsymbol{e}_1)(\tilde{\boldsymbol{c}} - \boldsymbol{e}_1)}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p}.$$

Note that

$$\begin{aligned} A(\tilde{\boldsymbol{c}}, p) &= \frac{|\tilde{\boldsymbol{c}}|^p - |\tilde{\boldsymbol{c}}|^{p-2}(\tilde{\boldsymbol{c}}, \boldsymbol{e}_1) - (\tilde{\boldsymbol{c}}, \boldsymbol{e}_1) + 1}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p} \\ &\geq \frac{|\tilde{\boldsymbol{c}}|^p - |\tilde{\boldsymbol{c}}|^{p-1} - |\tilde{\boldsymbol{c}}| + 1}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p} = \frac{\Big(|\tilde{\boldsymbol{c}}|^{p-1} - 1\Big)\Big(|\tilde{\boldsymbol{c}}| - 1\Big)}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p}. \end{aligned}$$

The last ratio is non-negative for $\tilde{\boldsymbol{c}} \neq \boldsymbol{e}_1$ and vanishes only if $|\tilde{\boldsymbol{c}}| = 1$.

In the case when $|\tilde{\boldsymbol{c}}| = 1$ we have

$$A(\tilde{\boldsymbol{c}}, p) = \frac{|\tilde{\boldsymbol{c}}|^p - |\tilde{\boldsymbol{c}}|^{p-2}(\tilde{\boldsymbol{c}}, \boldsymbol{e}_1) - (\tilde{\boldsymbol{c}}, \boldsymbol{e}_1) + 1}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p} = \frac{2\Big(1 - (\tilde{\boldsymbol{c}}, \boldsymbol{e}_1)\Big)}{|\tilde{\boldsymbol{c}} - \boldsymbol{e}_1|^p} > 0 \quad \text{for } \tilde{\boldsymbol{c}} \neq \boldsymbol{e}_1.$$

Thus, $A(\tilde{\boldsymbol{c}}, p) > 0$ for $\tilde{\boldsymbol{c}} \neq \boldsymbol{e}_1$. Taking into account the fact that

$$A(\tilde{\boldsymbol{c}}, p) \longrightarrow 1 \quad \text{as} \quad |\tilde{\boldsymbol{c}}| \to \infty,$$

we conclude that establishing the inequality

$$\liminf_{\tilde{c}\to e_1} A(\tilde{\boldsymbol{c}}, p) > 0$$

will complete the proof of the Proposition. To this end, set

$$\tilde{\boldsymbol{c}} = \boldsymbol{e}_1 + t\tilde{\boldsymbol{v}}, \quad |\tilde{\boldsymbol{v}}| = 1, \; t > 0. \tag{2.65}$$

Let us investigate the asymptotic behaviour of $A(\tilde{\boldsymbol{e}}_1 + t\tilde{\boldsymbol{v}}, p)$ as $t \to 0$. Substitution of (2.65) into the formula for $A(\tilde{\boldsymbol{c}}, p)$ delivers

$$A(\boldsymbol{e}_1 + t\tilde{\boldsymbol{v}}, p) = \frac{\Big(|\boldsymbol{e}_1 + t\tilde{\boldsymbol{v}}|^{p-2}(\boldsymbol{e}_1 + t\tilde{\boldsymbol{v}}) - \boldsymbol{e}_1\Big)t\tilde{\boldsymbol{v}}}{t^p}$$

$$= \Big((p-2)(\boldsymbol{e}_1, \tilde{\boldsymbol{v}})^2 + 1\Big)t^{2-p} + O\Big(t^{3-p}\Big) \geq \frac{1}{2}t^{2-p}$$

for sufficiently small $t > 0$, uniformly with respect to $\tilde{\boldsymbol{v}}$, $|\tilde{\boldsymbol{v}}| = 1$. Hence,

$$\liminf_{\tilde{c}\to e_1} A(\tilde{\boldsymbol{c}}, p) > 0. \quad \square$$

Chapter 3

Non-local homogenised limits for periodic composite media

Introduction

The mathematical theory of homogenisation establishes that in the classical case, *i.e.* when a periodic heterogeneous medium has a moderate contrast (mathematically, is described by uniformly elliptic PDE), the homogenised constitutive relations preserve the local character of the original heterogeneous constitutive relation. These relations are explicitly characterized in terms of solutions of certain canonical unit cell problems.

However, for a number of problems coming from applications these homogenised limits prove unable to explain *non-local* effects observed in the overall behaviour.

We rigorously establish that in the non-classical case, when a heterogeneous medium consists of materials with highly contrasting parameters, the homogenised constitutive relation may reveal a non-local structure. In particular, we show that for a class of homogenisation problems involving highly anisotropic fibres the homogenised equation is an integro-differential one, displaying a non-locality along the fibres. The kernel of the emerging integral operator is explicitly expressed in terms of the Green's function on the fibre, and the local part is determined as in the classical theory. This result is obtained using two different methods, two-scale convergence and asymptotic expansion.

There has been a number of works on non-local effects in mathematical homogenisation. They have mainly been concerned with homogenisation of dynamical problems and the non-localities observed in these works were of the memory-like type, *i.e.* the non-local parts of arising homogenised operators were convolution operators with respect to time t. In particular, Tartar [50] and Amirat *et al* [6] consider homogenisation of a rather general hyperbolic equation of first order and find the (weak) limit of the oscillating solutions as the small parameter tends to zero using Fourier and Laplace transforms. In his further work [51], Tartar rigorously proves that the limiting operator contains a convolution kernel with respect to time. Homogenisation of a general class of parabolic equations with memory has been established rigorously in works Fenchenko

& Khruslov [22] and Khruslov [28].

The above general results have been developed further by Allaire [4] and Zikov [60], who consider the problems of homogenisation of an elliptic equation and a parabolic equation, respectively, with coefficients that diminish as ε^γ, $\gamma > 0$ when $\varepsilon \to 0$. They have shown that the time non-locality arises in the critical case $\gamma = 2$ (the so-called double porosity case) and have derived the corresponding coupled system of homogenised equations.

The novel contribution of the work presented in this chapter is that we present a homogenisation problem whose homogenised limit exhibits non-locality in a spatial variable in the stationary setting. In the course of the study we rigorously prove the related convergence results and show the structure of the corresponding non-local constitutive relation.

Our approach uses a combination of methods previously studied in the literature: the method of two-scale convergence, introduced by Nguetseng [35] and further developed by Allaire [4], and the classical method of double-scale asymptotic expansions proved to be applicable to double-porosity problems in works of Sandrakov [39], [40].

3.1 Statement of the problem and its homogenisation

Formulation of the problem

We study the problem of homogenisation of composites consisting of two materials, one of which is included in the shape of fibres into the other, main material. The fibres are considered to be "hard" in one direction and "soft" in the orthogonal ones. The main material is hard in all directions. (To simplify analysis, we consider a 3D scalar (conductivity) version of this situation.)

Let us turn to the mathematical formulation of this problem. First we introduce some notation that proves to be useful in this chapter. By Q_2 we denote the two-dimensional unit cell $Q_2 = [0,1]^2$, and for a Q_2-periodic set $\tilde{F}_0 \subset \mathbf{R}^2$ with Lipschitz boundary, we consider the sets $F_0 := \tilde{F}_0 \times \mathbf{R}$, $\tilde{F}_1 := \mathbf{R}^2 \setminus \tilde{F}_0$, $F_1 := \mathbf{R}^3 \setminus F_0$. Throughout the chapter unless otherwise stated, we denote the volume fractions $|F_0 \cap Q|$ and $|F_1 \cap Q|$ of the phases F_0 and F_1 by $\mathfrak{f}_0$ and $\mathfrak{f}_1$, respectively. Note that $\mathfrak{f}_0 = |\tilde{F}_0 \cap Q_2|$ and $\mathfrak{f}_1 = |\tilde{F}_1 \cap Q_2|$. As before, we use the notation $Q := Q_2 \times [0,1]$. Furthermore, we fix a positive number T and denote $\mathbf{T} := [-T,T]^3$. For a small positive value ε such that $\varepsilon^{-1}T =: N$ is a natural number, we introduce contracted sets $F_0^\varepsilon := \varepsilon F_0$ and $F_1^\varepsilon := \varepsilon F_1$. Henceforth, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_2)$ denote points of $\mathbf{R}^3$ and Q respectively, and by $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{x}}$ we denote two-dimensional vectors (y_1, y_2) and (x_1, x_2) respectively.

We assume that the set F_1 is connected and $\mathfrak{f}_1 > 0$. Note that under these conditions the measure $\mathrm{d}\mu_1 := \mathrm{d}\boldsymbol{x}|_{F_1}$ (which is assumed to be zero outside the set F_1) is ergodic[1]

[1] A measure μ is called ergodic (on the period torus generated by Q) if $u = const$ μ-almost everywhere once there exist $u_n \in C^\infty_{per}(Q)$ such that $\int_Q |u_n - u|^2 \mathrm{d}\mu \to 0$ and $\int_Q |\nabla u_n|^2 \mathrm{d}\mu \to 0$ as $n \to \infty$.

and non-degenerate[2], which allows us to use certain techniques originated by Zhikov [60].

Define a matrix function $\left(A^{\varepsilon}_{ij}(\boldsymbol{y})\right)$ by the following formula

$$\left(A^{\varepsilon}_{ij}(\boldsymbol{y})\right) = \begin{cases} \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_1. \end{cases} \tag{3.1}$$

Consider an elliptic equation as follows

$$-\left(A^{\varepsilon}_{ij}(\boldsymbol{x}/\varepsilon)u_{,j}\right)_{,i} + \lambda u = f(\boldsymbol{x}), \quad \lambda \geq 0, \tag{3.2}$$

where the function $f(\boldsymbol{x})$ is infinitely smooth in $\mathbf{R}^3$ and $\mathbf{T}$-periodic. In the case $\lambda = 0$ we also assume that $\int_{\mathbf{T}} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$, and $\int_{\mathbf{T}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$.

The equation (3.2) is understood in the weak sense, that is, a $\mathbf{T}$-periodic function $u^{\varepsilon} \in H^1_{per}(\mathbf{T})$ is called a solution to the equation (3.2) if and only if for any function $\psi \in H^1_{per}(\mathbf{T})$ the following identity holds

$$\int\limits_{\mathbf{T}\cap F_1^{\varepsilon}} \nabla u^{\varepsilon} \nabla \psi \mathrm{d}\boldsymbol{x} + \varepsilon^2 \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} \left(u^{\varepsilon}_{,1}\psi_{,1} + u^{\varepsilon}_{,2}\psi_{,2}\right)\mathrm{d}\boldsymbol{x}$$

$$+ \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} u^{\varepsilon}_{,3}\psi_{,3}\mathrm{d}\boldsymbol{x} + \lambda \int\limits_{\mathbf{T}} u^{\varepsilon}\psi \mathrm{d}\boldsymbol{x} = \int\limits_{\mathbf{T}} f\psi \mathrm{d}\boldsymbol{x}. \tag{3.3}$$

We consider initially $\lambda > 0$ in (3.2) which allows to simplify certain analysis. The case $\lambda = 0$ is also considered and requires development of a certain Poincaré type inequality for high-contrast media, which has been implemented below (see Section 3.2).

Let us give the reader an informal flavour of the idea that led us to considering equation (3.2). In his paper [60], Zhikov considered the Cauchy problem for a parabolic equation in 2D as follows

$$\frac{\partial u^{\varepsilon}}{\partial t}(\tilde{\boldsymbol{x}}, t) - \left(a^{\varepsilon}_{ij}(\tilde{\boldsymbol{x}}/\varepsilon)u^{\varepsilon}_{,i}(\tilde{\boldsymbol{x}}, t)\right)_{,j} = 0, \quad \tilde{\boldsymbol{x}} \in \Omega \subset \mathbf{R}^2, \quad t > 0 \tag{3.4}$$

$$u^{\varepsilon}(\tilde{\boldsymbol{x}}, 0) = f(\tilde{\boldsymbol{x}}). \tag{3.5}$$

[2]A measure μ is said to be non-degenerate if zero vector is the only constant potential vector in $[L^2(Q, \mathrm{d}\mu)]^3$. (For the definition of a potential vector, see footnote in the Appendix C.)

Here, the entries of the 2×2 matrix $\left(a^{\varepsilon}_{ij}(\tilde{\boldsymbol{y}})\right)$ are defined by the formula

$$a^{\varepsilon}_{ij}(\tilde{\boldsymbol{y}}) = \left(\chi_{\tilde{F}_1}(\tilde{\boldsymbol{y}}) + \varepsilon^2 \chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}})\right)\delta_{ij},$$

where $\chi_{\tilde{F}_1}$ and $\chi_{\tilde{F}_0}$ are the characteristic functions of the sets $\tilde{F}_1$ and $\tilde{F}_0$, respectively. Zhikov [60] has shown that if Ω is a bounded domain with Lipschitz boundary then the solution $u^{\varepsilon} \in H^1_0(\Omega)$ of the problem (3.4)–(3.5) two-scale converges to the sum $u^{(1)}(\tilde{\boldsymbol{x}}, t) + v(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, t)$, where the functions $u^{(1)} \in H^1_0(\Omega)$ and $v \in L^2\left(\Omega, H^1_{per}(Q_2)\right)$ satisfy the following coupled system of equations

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial u^{(1)}}{\partial t} - \Delta_{\tilde{y}} v = 0, \quad \tilde{\boldsymbol{y}} \in \tilde{F}_0 \cap Q_2, \quad v|_{\tilde{y} \in \tilde{F}_1 \cap Q_2} = 0 \\ \frac{\partial u^{(1)}}{\partial t} + \frac{\partial \langle v \rangle}{\partial t} - \operatorname{div}\left(A^{hom}_{2D} \nabla u^{(1)}\right) = 0, \quad \tilde{\boldsymbol{x}} \in \Omega, \end{cases} \tag{3.6}$$

together with the initial conditions

$$u^{(1)}(\tilde{\boldsymbol{x}}, 0) = f(\tilde{\boldsymbol{x}}), \qquad v(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, 0) = 0.$$

Here, the two-dimensional homogenised matrix A^{hom}_{2D} is given by the following formula

$$A^{hom}_{2D} = \begin{pmatrix} \int\limits_{\tilde{F}_1 \cap Q_2} \left(1 + (N_1)_{,1}(\tilde{\boldsymbol{y}})\right) \mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1 \cap Q_2} (N_1)_{,2}(\tilde{\boldsymbol{y}}) \mathrm{d}\tilde{\boldsymbol{y}} \\ \int\limits_{\tilde{F}_1 \cap Q_2} (N_2)_{,1}(\tilde{\boldsymbol{y}}) \mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1 \cap Q_2} \left(1 + (N_2)_{,2}(\tilde{\boldsymbol{y}})\right) \mathrm{d}\tilde{\boldsymbol{y}} \end{pmatrix}, \tag{3.7}$$

where $N_1(\tilde{\boldsymbol{y}})$ and $N_2(\tilde{\boldsymbol{y}})$ are the solutions of the corresponding two-dimensional "unit-cell" problems in $\tilde{F}_1 \cap Q_2$ with Neumann boundary condition on $\partial \tilde{F}_1 \cap Q_2$ (see (3.89)).

Notice that by expressing the function v in terms of the function $u^{(1)}$ from the first of the equations (3.6) and substituting it in the second one, we obtain a non-local problem for $u^{(1)}$ containing a convolution operator with respect to time t.

We take Laplace transform of the problem (3.4) with respect to the variable t as follows

$$\hat{u}^{\varepsilon}(\tilde{\boldsymbol{x}}, \mu) := \int\limits_0^{\infty} u^{\varepsilon}(\tilde{\boldsymbol{x}}, t) \exp(-\mu t) \mathrm{d}t, \quad \mu > 0,$$

and arrive at the following spectral problem

$$-\left(a^{\varepsilon}_{ij}(\tilde{\boldsymbol{x}}/\varepsilon) \hat{u}^{\varepsilon}_{,i}(\tilde{\boldsymbol{x}})\right)_{,j} + \mu \hat{u}^{\varepsilon}(\tilde{\boldsymbol{x}}) = 0, \quad \mu > 0, \quad \tilde{\boldsymbol{x}} \in \Omega \subset \mathbf{R}^2. \tag{3.8}$$

If we now consider the case $\Omega = [-T, T]^2$ and introduce an additional spatial variable $x_3 \in [-T, T]$, so that the function $\hat{u}^{\varepsilon}(\boldsymbol{x})$ is sought to satisfy periodic boundary conditions with respect to $\boldsymbol{x} \in \mathbf{T}$, then the equation (3.8) can be formally viewed as the Fourier transform with respect to x_3 of the problem (3.2) with $\lambda = 0$ and $f(\boldsymbol{x}) = 0$. Hence formally, the parabolic equation (3.4) and the elliptic equation (3.2) turn out to be the inverse Laplace and Fourier transforms of the same equation (3.8).

From this point of view, time t in (3.4) is somewhat analogous to the spatial coordinate x_3 in (3.2). One can expect therefore that the time non-locality for the system (3.6) should translate to the spatial non-locality with respect to x_3 when passing to the limit $\varepsilon \to 0$ in the equation (3.2). This is indeed the case, as established rigorously in our work. Note however that the methods we develop in this chapter are more powerful, *i.e.* they are applicable to situations where no Laplace-Fourier transform analogy holds. As an illustration of this, consider an example where the matrix $\left(A^\varepsilon_{ij}(\boldsymbol{y})\right)$ is defined by the formula (*cf.* (3.1))

$$\left(A^\varepsilon_{ij}(\boldsymbol{y})\right) = \begin{cases} \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \alpha \end{pmatrix} & \text{if } \boldsymbol{y} \in F_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } \boldsymbol{y} \in F_1. \end{cases}$$

with positive $\alpha \neq 1$. It is obvious that in this case the Fourier transform is not applicable, yet the techniques of this chapter work successfully.

3.1.1 Passing to the limit in the equation (3.2) when $\varepsilon \to 0$

Using Lax-Milgram lemma (see *e.g.* Yosida [58]; Jikov *et al* [26]), it is not difficult to see that there exists a unique solution u^ε to the equation (3.2) in the class $H^1_{per}(\mathbf{T})$.

To find the limiting, or homogenised, behaviour of the solution u^ε when $\varepsilon \to 0$ we are going to implement the method of two-scale convergence, originated by Nguetseng [35] and further developed by Allaire [4] and Zhikov [60]. Basic facts about the two-scale convergence are reviewed in Appendix A. Let us first prove that the expressions u^ε and $\varepsilon\nabla u^\varepsilon$ are bounded in $L^2(\mathbf{T})$, uniformly with respect to $\varepsilon \in (0, 1]$, *i.e.*,

$$\int_{\mathbf{T}} |u^\varepsilon|^2 \mathrm{d}\boldsymbol{x} \leq C, \quad \varepsilon^2 \int_{\mathbf{T}} |\nabla u^\varepsilon|^2 \mathrm{d}\boldsymbol{x} \leq C, \tag{3.9}$$

where C is a constant independent of ε but depending, possibly, on λ. Taking $\psi = u^\varepsilon$ in (3.3) we get

$$\int_{\mathbf{T}\cap F_1^\varepsilon} |\nabla u^\varepsilon|^2 \mathrm{d}\boldsymbol{x} + \varepsilon^2 \int_{\mathbf{T}\cap F_0^\varepsilon} \left(\left(u^\varepsilon_{,1}\right)^2 + \left(u^\varepsilon_{,2}\right)^2 \right) \mathrm{d}\boldsymbol{x}$$

$$+ \int_{\mathbf{T}\cap F_0^\varepsilon} \left(u^\varepsilon_{,3}\right)^2 \mathrm{d}\boldsymbol{x} + \lambda \int_{\mathbf{T}} (u^\varepsilon)^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} f u^\varepsilon \mathrm{d}\boldsymbol{x}.$$

Hence,

$$\int\limits_{\mathbf{T}\cap F_1^\varepsilon} |\nabla u^\varepsilon|^2 \mathrm{d}\boldsymbol{x} + \varepsilon^2 \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \left(\left(u^\varepsilon_{,1}\right)^2 + \left(u^\varepsilon_{,2}\right)^2 \right) \mathrm{d}\boldsymbol{x} + \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \left(u^\varepsilon_{,3}\right)^2 \mathrm{d}\boldsymbol{x} + \lambda \int\limits_{\mathbf{T}} (u^\varepsilon)^2 \,\mathrm{d}\boldsymbol{x}$$

$$\leq \left(\int\limits_{\mathbf{T}} f^2 \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \left(\int\limits_{\mathbf{T}} (u^\varepsilon)^2 \,\mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \leq \frac{1}{2\lambda} \int\limits_{\mathbf{T}} f^2 \mathrm{d}\boldsymbol{x} + \frac{\lambda}{2} \int\limits_{\mathbf{T}} (u^\varepsilon)^2 \,\mathrm{d}\boldsymbol{x},$$

and, therefore,

$$\int\limits_{\mathbf{T}\cap F_1^\varepsilon} |\nabla u^\varepsilon|^2 \mathrm{d}\boldsymbol{x} + \varepsilon^2 \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \left(\left(u^\varepsilon_{,1}\right)^2 + \left(u^\varepsilon_{,2}\right)^2 \right) \mathrm{d}\boldsymbol{x}$$

$$+ \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \left(u^\varepsilon_{,3}\right)^2 \mathrm{d}\boldsymbol{x} + \frac{\lambda}{2} \int\limits_{\mathbf{T}} (u^\varepsilon)^2 \,\mathrm{d}\boldsymbol{x} \leq \frac{1}{2\lambda} \int\limits_{\mathbf{T}} f^2 \mathrm{d}\boldsymbol{x}. \tag{3.10}$$

Thus, we obtain the *a priori* bounds (3.9).

Using the compactness property of two-scale convergence (see *e.g.* Nguetseng [35]; Allaire [4]; Appendix A), we deduce that up to a subsequence

$$u^\varepsilon(\boldsymbol{x}) \xrightarrow{2} u(\boldsymbol{x}, \boldsymbol{y}) \in L^2\Big(\mathbf{T}, H^1_{per}(Q)\Big) \tag{3.11}$$

and

$$\varepsilon \nabla u^\varepsilon(\boldsymbol{x}) \xrightarrow{2} \nabla_y u(\boldsymbol{x}, \boldsymbol{y}). \tag{3.12}$$

Note in passing that the latter convergence statement implies that denoting the characteristic function of the set F_1 by $\chi_1(\boldsymbol{y})$ we have

$$\varepsilon \chi_1(\varepsilon^{-1}\boldsymbol{x}) \nabla u^\varepsilon(\boldsymbol{x}) \xrightarrow{2} \chi_1(\boldsymbol{y}) \nabla_y u(\boldsymbol{x}, \boldsymbol{y}). \tag{3.13}$$

Here we use one of the basic properties of two-scale convergence (see *e.g.* Zhikov [60]; Appendix A).

On the other hand, it is easy to see from (3.10) that the expression $\chi_1(\varepsilon^{-1}\boldsymbol{x})\nabla u^\varepsilon(\boldsymbol{x})$ is bounded in $L^2(\mathbf{T})$ and therefore $\varepsilon\chi_1(\varepsilon^{-1}\boldsymbol{x})\nabla u^\varepsilon(\boldsymbol{x}) \longrightarrow 0$ in $\left[L^2(\mathbf{T})\right]^3$ strong. In particular,

$$\varepsilon \chi_1(\varepsilon^{-1}\boldsymbol{x}) \nabla u^\varepsilon(\boldsymbol{x}) \xrightarrow{2} 0. \tag{3.14}$$

Comparing (3.13) and (3.14) we conclude that

$$\chi_1(\boldsymbol{y}) \nabla_y u(\boldsymbol{x}, \boldsymbol{y}) = \nabla_y u(\boldsymbol{x}, \boldsymbol{y})|_{y \in F_1 \cap Q} = 0.$$

Therefore, $u(\boldsymbol{x}, \boldsymbol{y})|_{y\in F_1\cap Q} = u^{(1)}(\boldsymbol{x})$ for some function $u^{(1)}(\boldsymbol{x}) \in L^2(\mathbf{T})$. Moreover, in Appendix B we prove that $u^{(1)}(\boldsymbol{x}) \in H^1_{per}(\mathbf{T})$.

In the same fashion $\varepsilon u^\varepsilon_{,3}(\boldsymbol{x}) \longrightarrow 0$ in $L^2(\mathbf{T})$ and hence $u_{,y_3}(\boldsymbol{x}, \boldsymbol{y}) = 0$.

Thus, $u \in V$, where the functional space V is defined by the following formula

$$V = \Big\{ u(\boldsymbol{x}, \tilde{\boldsymbol{y}}) \in L^2\Big(\mathbf{T}, H^1_{per}(Q_2)\Big), u|_{\tilde{y} \in \tilde{F}_1 \cap Q_2} = u^{(1)}(\boldsymbol{x}) \in H^1_{per}(\mathbf{T}), \nabla_{\tilde{y}} u|_{\tilde{y} \in \tilde{F}_1 \cap Q_2} = 0 \Big\}.$$

Let us consider test functions ψ of a particular type $\psi(\boldsymbol{x}) = \psi^\varepsilon(\boldsymbol{x}) = \Phi(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}})$, where $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ is of the form

$$\Phi(\boldsymbol{x}, \tilde{\boldsymbol{y}}) = \Phi_1(\boldsymbol{x}) + \alpha(\boldsymbol{x}) h(\tilde{\boldsymbol{y}}). \tag{3.15}$$

Here $\Phi_1(\boldsymbol{x}), \alpha(\boldsymbol{x}) \in C^\infty_{per}(\mathbf{T})$, and $h \in X := \Big\{ h(\tilde{\boldsymbol{y}}) \in C^\infty_{per}(Q_2), h|_{\tilde{F}_1 \cap Q_2} = 0 \Big\}$. Obviously,

$$\nabla \psi^\varepsilon(\boldsymbol{x})|_{\mathbf{T} \cap F_1^\varepsilon} = \nabla \Phi_1(\boldsymbol{x})|_{\mathbf{T} \cap F_1^\varepsilon}$$

and

$$\begin{aligned} \varepsilon \nabla \psi^\varepsilon(\boldsymbol{x}) &= \varepsilon \nabla \Phi_1(\boldsymbol{x}) + \varepsilon \nabla \alpha(\boldsymbol{x}) h(\varepsilon^{-1}\tilde{\boldsymbol{x}}) + \alpha(\boldsymbol{x}) \nabla_y h(\tilde{\boldsymbol{y}})|_{y=\varepsilon^{-1}x} \\ &= \alpha(\boldsymbol{x}) \nabla_y h(\tilde{\boldsymbol{y}})|_{y=\varepsilon^{-1}x} + o(1), \end{aligned}$$

where $o(1)$ is understood in the sense of $L^2(\mathbf{T})$-norm.

Substituting $\psi = \psi^\varepsilon$ into the identity (3.3) we get

$$\begin{aligned} &\int_{\mathbf{T} \cap F_1^\varepsilon} \nabla u^\varepsilon(\boldsymbol{x}) \nabla \Phi_1(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &+\varepsilon \int_{\mathbf{T} \cap F_0^\varepsilon} \Big(u^\varepsilon_{,1}(\boldsymbol{x}) \alpha(\boldsymbol{x}) h_{,y_1}(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}x} + u^\varepsilon_{,2}(\boldsymbol{x}) \alpha(\boldsymbol{x}) h_{,y_2}(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}x} \Big) \mathrm{d}\boldsymbol{x} \\ &+\varepsilon \int_{\mathbf{T} \cap F_0^\varepsilon} \left(u^\varepsilon_{,1}(\boldsymbol{x}) o(1) + u^\varepsilon_{,2}(\boldsymbol{x}) o(1) \right) \mathrm{d}\boldsymbol{x} + \int_{\mathbf{T} \cap F_0^\varepsilon} u^\varepsilon_{,3}(\boldsymbol{x}) \Phi_{,x_3}(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}}) \mathrm{d}\boldsymbol{x} \\ &+\lambda \int_{\mathbf{T}} u^\varepsilon(\boldsymbol{x}) \Phi(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}}) \mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} f(\boldsymbol{x}) \Phi(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}}) \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.16}$$

We want to pass to the limit when $\varepsilon \to 0$ in the identity (3.16). First, as is shown in the Appendix C,

$$\lim_{\varepsilon \to 0} \int_{\mathbf{T} \cap F_1^\varepsilon} \nabla u^\varepsilon(\boldsymbol{x}) \nabla \Phi_1(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} A_1^{hom} \nabla u^{(1)}(\boldsymbol{x}) \nabla \Phi_1(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{3.17}$$

where (*cf.* (3.7))

$$A_1^{hom} = \begin{pmatrix} \int\limits_{\tilde{F}_1 \cap Q_2} \Big(1 + (N_1)_{,1}(\tilde{\boldsymbol{y}})\Big) \mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1 \cap Q_2} (N_1)_{,2}(\tilde{\boldsymbol{y}}) \mathrm{d}\tilde{\boldsymbol{y}} & 0 \\ \int\limits_{\tilde{F}_1 \cap Q_2} (N_2)_{,1}(\tilde{\boldsymbol{y}}) \mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1 \cap Q_2} \Big(1 + (N_2)_{,2}(\tilde{\boldsymbol{y}})\Big) \mathrm{d}\tilde{\boldsymbol{y}} & 0 \\ 0 & 0 & \mathfrak{f}_1 \end{pmatrix}. \tag{3.18}$$

Secondly, using (3.12) we get

$$\varepsilon \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \Big(u^\varepsilon_{,1}(\boldsymbol{x})\alpha(\boldsymbol{x})h_{,1}(\tilde{\boldsymbol{y}})|_{\tilde{\boldsymbol{y}}=\varepsilon^{-1}\tilde{\boldsymbol{x}}} + u^\varepsilon_{,2}(\boldsymbol{x})\alpha(\boldsymbol{x})h_{,2}(\tilde{\boldsymbol{y}})|_{\tilde{\boldsymbol{y}}=\varepsilon^{-1}\tilde{\boldsymbol{x}}}\Big)\mathrm{d}\boldsymbol{x}$$

$$= \varepsilon \int\limits_{\mathbf{T}} \Big(u^\varepsilon_{,1}(\boldsymbol{x})\alpha(\boldsymbol{x})h_{,1}(\tilde{\boldsymbol{y}})\chi_0(\tilde{\boldsymbol{y}}) + u^\varepsilon_{,2}(\boldsymbol{x})\alpha(\boldsymbol{x})h_{,2}(\tilde{\boldsymbol{y}})\chi_0(\tilde{\boldsymbol{y}})\Big)\Big|_{\tilde{\boldsymbol{y}}=\varepsilon^{-1}\tilde{\boldsymbol{x}}}\mathrm{d}\boldsymbol{x}$$

$$\xrightarrow{\varepsilon\to 0} \int\limits_{\mathbf{T}}\int\limits_{Q_2} \Big(u_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})h_{,1}(\tilde{\boldsymbol{y}})\chi_0(\tilde{\boldsymbol{y}}) + u_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})h_{,2}(\tilde{\boldsymbol{y}})\chi_0(\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$= \int\limits_{\mathbf{T}}\int\limits_{\tilde{F}_0\cap Q_2} \Big(u_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})h_{,1}(\tilde{\boldsymbol{y}}) + u_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})h_{,2}(\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$= \int\limits_{\mathbf{T}}\int\limits_{\tilde{F}_0\cap Q_2} \Big(u_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}})\Phi_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}}) + u_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})\Phi_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}.$$

In the above formulas, χ_0 denotes the characteristic function of the set $\tilde{F}_0 \cap Q_2$.

Furthermore, due to the fact that $\varepsilon\nabla u^\varepsilon$ is bounded in $L^2(\mathbf{T})$, the following convergence holds

$$\varepsilon \int\limits_{\mathbf{T}\cap F_0^\varepsilon} \Big(u^\varepsilon_{,1}(\boldsymbol{x})o(1) + u^\varepsilon_{,2}(\boldsymbol{x})o(1)\Big)\mathrm{d}\boldsymbol{x} \xrightarrow{\varepsilon\to 0} 0.$$

Finally, we find the limit of the expression

$$\int\limits_{\mathbf{T}\cap F_0^\varepsilon} u^\varepsilon_{,3}(\boldsymbol{x})\Phi_{,x_3}(\boldsymbol{x},\varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x} \tag{3.19}$$

as $\varepsilon \to 0$. Notice that the inequality (3.10) implies the following estimate

$$\int\limits_{\mathbf{T}} \left(u^\varepsilon_{,3}\right)^2 \mathrm{d}\boldsymbol{x} \le C$$

and therefore, there exists such $q(\boldsymbol{x},\boldsymbol{y}) \in L^2(\mathbf{T}\times Q)$ that up to a subsequence,

$$u^\varepsilon_{,3} \xrightharpoonup{2} q(\boldsymbol{x},\boldsymbol{y}). \tag{3.20}$$

In the course of finding the limit of the expression (3.19) we establish the following lemma.

Lemma.

Let $u(\boldsymbol{x},\tilde{\boldsymbol{y}}) \in V$ and $q(\boldsymbol{x},\boldsymbol{y})$ be the limiting functions from (3.11) and (3.20). Then there exists the Sobolev derivative $u_{,x_3}(\boldsymbol{x},\tilde{\boldsymbol{y}}) \in L^2(\mathbf{T}\times Q_2)$ and it is given by the equality $u_{,x_3}(\boldsymbol{x},\tilde{\boldsymbol{y}}) = \Big\langle q(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3}$.

Proof:

Assume that $\Psi = \Psi(\boldsymbol{x}, \tilde{\boldsymbol{y}}) \in C^\infty_{per}\Big(\mathbf{T}, C^\infty_{per}(Q_2)\Big)$. By means of integration by parts we write the equality

$$\int\limits_{\mathbf{T}} u^\varepsilon_{,3}(\boldsymbol{x})\Psi(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x} = -\int\limits_{\mathbf{T}} u^\varepsilon(\boldsymbol{x})\Psi_{,x_3}(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x}.$$

Using convergences (3.11) and (3.20) we pass to the limit as $\varepsilon \to 0$ in the last equality and obtain the following identity

$$\int\limits_{\mathbf{T}}\int\limits_{Q} q(\boldsymbol{x}, \boldsymbol{y})\Psi(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} = -\int\limits_{\mathbf{T}}\int\limits_{Q_2} u(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Psi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}.$$

It can be rewritten in the following way

$$\int\limits_{\mathbf{T}}\int\limits_{Q_2} \Big\langle q(\boldsymbol{x}, \boldsymbol{y})\Big\rangle_{y_3} \Psi(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x} = -\int\limits_{\mathbf{T}}\int\limits_{Q_2} u(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Psi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}.$$

By the definition of Sobolev derivative, it implies the statement of the lemma. □

Now, using the above lemma we get

$$\int\limits_{\mathbf{T}\cap F_0^\varepsilon} u^\varepsilon_{,3}(\boldsymbol{x})\Psi(\boldsymbol{x}, \varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x} \overset{\varepsilon\to\infty}{\longrightarrow} \int\limits_{\mathbf{T}}\int\limits_{Q} q(\boldsymbol{x}, \boldsymbol{y})\Psi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}$$

$$= \int\limits_{\mathbf{T}}\int\limits_{Q_2} \Big\langle q(\boldsymbol{x}, \boldsymbol{y})\Big\rangle_{y_3} \Phi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x} = \int\limits_{\mathbf{T}}\int\limits_{Q_2} u_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Phi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}.$$

Using obtained convergence results, we pass to the limit in (3.16) as $\varepsilon \to 0$ and get the following identity

$$\int\limits_{\mathbf{T}} A_1^{hom}\nabla u^{(1)}(\boldsymbol{x})\nabla\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int\limits_{\mathbf{T}}\int\limits_{\tilde{F}_0\cap Q_2}\Big(u_{,y_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Phi_{,y_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}}) + u_{,y_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Phi_{,y_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$+ \int\limits_{\mathbf{T}}\int\limits_{Q_2} u_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Phi_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x} + \lambda\int\limits_{\mathbf{T}}\int\limits_{Q_2} u(\boldsymbol{x}, \tilde{\boldsymbol{y}})\Phi(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$= \int\limits_{\mathbf{T}}\int\limits_{Q} f(\boldsymbol{x})\Phi(\boldsymbol{x}, \tilde{\boldsymbol{y}})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} \tag{3.21}$$

for any function Φ of the form (3.15).

The identity (3.21) is naturally equivalent to a system of two partial differential equations. To see this, we take first $\Phi = \Phi_1(\boldsymbol{x}) \in C^\infty_{per}(\mathbf{T})$ in (3.21) and obtain the following equality

$$\int_{\mathbf{T}} A_1^{hom}\nabla u^{(1)}(\boldsymbol{x})\nabla\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_{\mathbf{T}}\int_{\tilde{F}_0\cap Q_2} u_{,x_3}(\boldsymbol{x},\tilde{\boldsymbol{y}})(\Phi_1)_{,x_3}(\boldsymbol{x})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$+\lambda\int_{\mathbf{T}}\int_{Q_2} u(\boldsymbol{x},\tilde{\boldsymbol{y}})\Phi_1(\boldsymbol{x})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}}\int_{Q} f(\boldsymbol{x})\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x},$$

that can be rewritten in the following way

$$\int_{\mathbf{T}} A_1^{hom}\nabla u^{(1)}(\boldsymbol{x})\nabla\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \mathfrak{f}_0\int_{\mathbf{T}} u_{,3}^{(1)}(\boldsymbol{x})(\Phi_1)_{,3}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

$$+\int_{\mathbf{T}}\langle w\rangle_{,3}(\boldsymbol{x})(\Phi_1)_{,3}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \lambda\int_{\mathbf{T}}\Big(u^{(1)}(\boldsymbol{x}) + \langle w\rangle(\boldsymbol{x})\Big)\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}} f(\boldsymbol{x})\Phi_1(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where $\langle w\rangle(\boldsymbol{x}) := \int_Q w(\boldsymbol{x},\tilde{\boldsymbol{y}})\mathrm{d}\boldsymbol{y}$. Note that the last identity holds for any $\Phi_1(\boldsymbol{x}) \in C_{per}^{\infty}(\mathbf{T})$. Therefore, it is equivalent to the following equation

$$-\mathrm{div}\Big(A^{hom}\nabla u^{(1)}\Big) - \frac{\partial^2\langle w\rangle}{\partial x_3^2} + \lambda\Big(u^{(1)} + \langle w\rangle\Big) = f, \tag{3.22}$$

where matrix A^{hom} is defined by the following formula (*cf.* (3.18), (3.7))

$$A^{hom} = \begin{pmatrix} \int\limits_{\tilde{F}_1\cap Q_2}\Big(1 + (N_1)_{,1}(\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1\cap Q_2}(N_1)_{,2}(\tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}} & 0 \\ \int\limits_{\tilde{F}_1\cap Q_2}(N_2)_{,1}(\tilde{\boldsymbol{y}})\mathrm{d}\tilde{\boldsymbol{y}} & \int\limits_{\tilde{F}_1\cap Q_2}\Big(1 + (N_2)_{,2}(\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, taking $\Phi(\boldsymbol{x},\tilde{\boldsymbol{y}}) = \alpha(\boldsymbol{x})h(\tilde{\boldsymbol{y}})$, where $\alpha(\boldsymbol{x}) \in C_{per}^{\infty}(\mathbf{T})$, and $h(\tilde{\boldsymbol{y}}) \in X$, and substituting it into the identity (3.21), we obtain the following equality

$$\int_{\mathbf{T}}\int_{\tilde{F}_0\cap Q_2}\Big(w_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}})h_{,1}(\tilde{\boldsymbol{y}}) + w_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})h_{,2}(\tilde{\boldsymbol{y}})\Big)\alpha(\boldsymbol{x})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$+\int_{\mathbf{T}}\int_{\tilde{F}_0\cap Q_2} u_{,x_3}(\boldsymbol{x},\tilde{\boldsymbol{y}})h(\tilde{\boldsymbol{y}})\alpha_{,3}(\boldsymbol{x})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x}$$

$$+\lambda\int_{\mathbf{T}}\int_{\tilde{F}_0\cap Q_2} u(\boldsymbol{x},\tilde{\boldsymbol{y}})h(\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})\mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\boldsymbol{x} = \int_{\mathbf{T}}\int_{Q} f(\boldsymbol{x})h(\tilde{\boldsymbol{y}})\alpha(\boldsymbol{x})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}.$$

The last identity holds for any $\alpha(\boldsymbol{x}) \in C_{per}^{\infty}(\mathbf{T})$ and $h(\tilde{\boldsymbol{y}}) \in C_{per}^{\infty}(Q_2)$, $h|_{\tilde{\boldsymbol{y}}\in\tilde{F}_1\cap Q_2} = 0$.

Hence, it is equivalent to the partial differential equation as follows

$$-\frac{\partial^2 w}{\partial y_1^2}-\frac{\partial^2 w}{\partial y_2^2}-\frac{\partial^2 w}{\partial x_3^2}-\frac{\partial^2 u^{(1)}}{\partial x_3^2}+\lambda\Big(u^{(1)}+w\Big)=f,\quad \tilde{\boldsymbol{y}}\in\tilde{F}_0\cap Q_2. \tag{3.23}$$

Summarising what we have got we conclude that

$$u^\varepsilon(\boldsymbol{x}) \xrightarrow{2} u^{(1)}(\boldsymbol{x})+w(\boldsymbol{x},\tilde{\boldsymbol{y}}) \tag{3.24}$$

and the $\mathbf{T}$-periodic (with respect to $\boldsymbol{x}$) functions $u^{(1)}$ and w satisfy the following system of elliptic equations

$$\begin{cases} -\mathrm{div}\Big(A^{hom}\nabla u^{(1)}\Big)-\dfrac{\partial^2\langle w\rangle}{\partial x_3^2}+\lambda\Big(u^{(1)}+\langle w\rangle\Big)=f, & \boldsymbol{x}\in\mathbf{T},\\ -\dfrac{\partial^2 w}{\partial y_1^2}-\dfrac{\partial^2 w}{\partial y_2^2}-\dfrac{\partial^2 w}{\partial x_3^2}-\dfrac{\partial^2 u^{(1)}}{\partial x_3^2}+\lambda\Big(u^{(1)}+w\Big)=f, & \tilde{\boldsymbol{y}}\in\tilde{F}_0\cap Q_2, \end{cases} \tag{3.25}$$

together with the boundary condition

$$w(\boldsymbol{x},\tilde{\boldsymbol{y}})\Big|_{\tilde{\boldsymbol{y}}\in\tilde{F}_0\cap Q_2}=0. \tag{3.26}$$

3.2 Remark on the case $\lambda=0$

Before proceeding further in our study of the homogenised system (3.25), let us make an important aside and note that in the original equation (3.2) the parameter λ can be set to zero[3] and this does not alter the validity of our results, in particular, in this case the homogenised equation (3.25) with $\lambda=0$ holds.[4] This follows from the next theorem.

Theorem.

Let $\mathbf{T}=[-T,T]^d$, $T>0$; $Q=[0,1]^d$; F_0 is a Q-periodic set with Lipschitz boundary, $F_1=\mathbf{R}^d\setminus F_0$ is connected; $F_0^\varepsilon=\varepsilon F_0$, $F_1^\varepsilon=\varepsilon F_1$, where $\varepsilon>0$ is such that $\varepsilon^{-1}T=:N$ is a natural number.

Then for any function $u\in H^1(\mathbf{T})$ with zero mean over $\mathbf{T}$ the following Poincaré-type inequality holds

$$\|u\|_{L^2(\mathbf{T})}\le C\Big(\|\nabla u\|_{L^2(\mathbf{T}\cap F_1^\varepsilon)}+\varepsilon\|\nabla u\|_{L^2(\mathbf{T}\cap F_0^\varepsilon)}\Big), \tag{3.27}$$

where the constant C does not depend on ε.

Proof:

[3] Note that in this case the solution $u(\boldsymbol{x})$ is sought in the class $H^1_{0,per}$ of functions having zero mean over $\mathbf{T}$.

[4] Of course, in the case $\lambda=0$ in addition to the condition (3.26) one more constraint should be imposed on the unknown functions $u^{(1)}$ and w, namely $\int_{\mathbf{T}}(u^{(1)}(\boldsymbol{x})+\langle w\rangle(\boldsymbol{x}))d\boldsymbol{x}=0$. We get this condition by observing that $\int_{\mathbf{T}}u^\varepsilon(\boldsymbol{x})d\boldsymbol{x}\to\int_{\mathbf{T}}\int_Q(u_1(\boldsymbol{x})+w(\boldsymbol{x},\tilde{\boldsymbol{y}}))d\boldsymbol{y}d\boldsymbol{x}$ due to the two-scale convergence (3.24).

In the proof we use some ideas presented in the paper by Allaire and Murat [5].

Throughout the proof we use the following notation $\langle g\rangle_D := |D|^{-1}\int_D g(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, where D is a bounded measurable set in $\mathbf{R}^d$ and function g belongs to the space $L^1(D)$. Let us note some properties of the operation of averaging $\langle\cdot\rangle_D$ that will prove useful to us. First, this operation is linear, *i.e.* for any $g_1, g_2 \in L^1(D)$

$$\langle g_1 + g_2\rangle_D = \langle g_1\rangle_D + \langle g_2\rangle_D. \tag{3.28}$$

Furthermore, if $D = D_0 \cup D_1$ is a partition of the set D into two disjoint subsets D_0 and D_1, and values $\mathfrak{f}_0 = |D_0|^{-1}|D|$ and $\mathfrak{f}_1 = |D_1|^{-1}|D|$ are their volume fractions, then

$$\langle g\rangle_D = \mathfrak{f}_0\langle g\rangle_{D_0} + \mathfrak{f}_1\langle g\rangle_{D_1}. \tag{3.29}$$

Finally, if $g \in L^2(D)$ then the following inequality is true

$$\langle g^2\rangle \geq \langle g\rangle^2. \tag{3.30}$$

One of many possible ways of proving it is by using orthogonality of the functions $g - \langle g\rangle$ and g as elements of $L^2(D)$.

Now, partition the cube $\mathbf{T}$ into disjoint cubes Q_ε^i, $i = 1, ..., K_\varepsilon$ of size ε, where $K_\varepsilon := N^d$, and consider two piecewise constant functions on $\mathbf{T}$ as follows

$$\bar{u}(\boldsymbol{x}) = \langle u\rangle_{Q_\varepsilon^i},\ \boldsymbol{x} \in Q_\varepsilon^i$$

and

$$\tilde{u}(\boldsymbol{x}) = \langle u\rangle_{Q_\varepsilon^i \cap F_1^\varepsilon},\ \boldsymbol{x} \in Q_\varepsilon^i.$$

Using the triangle inequality in $L^2(\mathbf{T})$ we get

$$\|u\|_{L^2(\mathbf{T})} \leq \|u - \bar{u}\|_{L^2(\mathbf{T})} + \|\bar{u} - \tilde{u}\|_{L^2(\mathbf{T})} + \|\tilde{u}\|_{L^2(\mathbf{T})}. \tag{3.31}$$

Let us estimate each of the terms coming into (3.31).

$$\|u - \bar{u}\|^2_{L^2(\mathbf{T})} = \sum_{i=1}^{K_\varepsilon}\int_{Q_\varepsilon^i}\left|u(\boldsymbol{x}) - \langle u\rangle_{Q_\varepsilon^i}\right|^2\mathrm{d}\boldsymbol{x}$$

$$= \varepsilon^d\sum_{i=1}^{K_\varepsilon}\int_{\varepsilon^{-1}Q^i\varepsilon}\left|u(\varepsilon\boldsymbol{x}') - \Big\langle u(\varepsilon\boldsymbol{x}')\Big\rangle_{\boldsymbol{x}'\in\varepsilon^{-1}Q_\varepsilon^i}\right|^2\mathrm{d}\boldsymbol{x}' \leq \varepsilon^d C_P(Q)\sum_{i=1}^{K_\varepsilon}\int_{\varepsilon^{-1}Q_\varepsilon^i}\left|\nabla_{\boldsymbol{x}'}u(\varepsilon\boldsymbol{x}')\right|^2\mathrm{d}\boldsymbol{x}'$$

$$= \varepsilon^2 C_P(Q)\sum_{i=1}^{K_\varepsilon}\int_{Q_\varepsilon^i}\left|\nabla u(\boldsymbol{x})\right|^2\mathrm{d}\boldsymbol{x} = \varepsilon^2 C_P(Q)\|\nabla u\|_{L^2(\mathbf{T})}.$$

Here we split the original integral into a sum of K_ε constituent integrals, then rescale the variable of integration, after that use the Poincaré inequality (3.39) for each cell $\varepsilon^{-1}Q^i_\varepsilon$, and finally rescale the variable of integration back.

Thus,

$$\|u - \bar{u}\|_{L^2(\mathbf{T})} \leq C_1\varepsilon\|\nabla u\|_{L^2(\mathbf{T})}, \tag{3.32}$$

where $C_1 = \sqrt{C_P(Q)}$.

Lemma.

Let Ω be a bounded connected domain in $\mathbf{R}^d$ with Lipschitz boundary. Then for any partition $\Omega = \Omega_0 \cup \Omega_1$ of the domain Ω into two disjoint subsets Ω_0 and Ω_1 with non-zero volumes there exists a positive constant $C = C(\Omega_0, \Omega_1)$ such that for any function $v \in H^1(\Omega)$ the following inequality holds

$$\Big|\langle v\rangle_{\Omega_0} - \langle v\rangle_{\Omega_1}\Big| \leq C\|\nabla v\|_{L^2(\Omega)}. \tag{3.33}$$

Proof:

Note first that if we denote the volume fractions $|\Omega_0|^{-1}|\Omega|$ and $|\Omega_1|^{-1}|\Omega|$ of the constituent subsets Ω_0 and Ω_1 by $\mathfrak{f}_0$ and $\mathfrak{f}_1$ respectively, then

$$\Big\langle|v - \langle v\rangle_\Omega|^2\Big\rangle_\Omega = \mathfrak{f}_0\Big\langle|v - \langle v\rangle_\Omega|^2\Big\rangle_{\Omega_0} + \mathfrak{f}_1\Big\langle|v - \langle v\rangle_\Omega|^2\Big\rangle_{\Omega_1} \tag{3.34}$$

$$\geq \mathfrak{f}_0\Big\langle|v - \langle v\rangle_\Omega|\Big\rangle^2_{\Omega_0} + \mathfrak{f}_1\Big\langle|v - \langle v\rangle_\Omega|\Big\rangle^2_{\Omega_1} \geq \mathfrak{f}_0\Big\langle v - \langle v\rangle_\Omega\Big\rangle^2_{\Omega_0} + \mathfrak{f}_1\Big\langle v - \langle v\rangle_\Omega\Big\rangle^2_{\Omega_1} \tag{3.35}$$

$$= \mathfrak{f}_0\Big(\langle v\rangle_{\Omega_0} - \langle v\rangle_\Omega\Big)^2 + \mathfrak{f}_1\Big(\langle v\rangle_{\Omega_1} - \langle v\rangle_\Omega\Big)^2 \tag{3.36}$$

$$= \mathfrak{f}_0\mathfrak{f}_1^2\Big(\langle v\rangle_{\Omega_0} - \langle v\rangle_{\Omega_1}\Big)^2 + \mathfrak{f}_1\mathfrak{f}_0^2\Big(\langle v\rangle_{\Omega_1} - \langle v\rangle_{\Omega_0}\Big)^2 = \mathfrak{f}_0\mathfrak{f}_1\Big(\langle v\rangle_{\Omega_0} - \langle v\rangle_{\Omega_1}\Big)^2. \tag{3.37}$$

In (3.34),(3.35) and (3.36) we use properties (3.29), (3.28) and (3.30) of the operation $\langle\cdot\rangle_D$ with $D = \Omega$, respectively. After that, in (3.37) we use the property (3.29) again, and then the trivial fact that $\mathfrak{f}_0 + \mathfrak{f}_1 = 1$. Thus, it is proved that

$$\Big(\langle v\rangle_{\Omega_0} - \langle v\rangle_{\Omega_1}\Big)^2 \leq (\mathfrak{f}_0\mathfrak{f}_1)^{-1}\Big\langle|v - \langle v\rangle_\Omega|^2\Big\rangle_\Omega. \tag{3.38}$$

Now, using the formula (3.38) and the Poincaré inequality as follows

$$\Big\langle|v - \langle v\rangle_\Omega|^2\Big\rangle_\Omega \leq C_P(\Omega)\|\nabla v\|^2_{L^2(\Omega)} \tag{3.39}$$

we get

$$\Big(\langle v\rangle_{\Omega_0} - \langle v\rangle_{\Omega_1}\Big)^2 \leq (\mathfrak{f}_0\mathfrak{f}_1)^{-1}C_P(\Omega)\|\nabla v\|^2_{L^2(\Omega)}, \tag{3.40}$$

which easily implies (3.33), where $C = \sqrt{(\mathfrak{f}_0\mathfrak{f}_1)^{-1}C_P(\Omega)}$. □

The second term in the right hand side of (3.31) is now estimated as follows

$$\|\bar{u}-\tilde{u}\|_{L^2(\mathbf{T})}^2 = \sum_{i=1}^{K_\varepsilon} \int\limits_{Q_\varepsilon^i} \Big(\langle u \rangle_{Q_\varepsilon^i} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big)^2 \mathrm{d}\boldsymbol{x} \tag{3.41}$$

$$= \varepsilon^d \sum_{i=1}^{K_\varepsilon} \Big(\langle u \rangle_{Q_\varepsilon^i} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big)^2 = \varepsilon^d \mathfrak{f}_0^2 \sum_{i=1}^{K_\varepsilon} \Big(\langle u \rangle_{Q_\varepsilon^i \cap F_0^\varepsilon} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big)^2 \tag{3.42}$$

$$= \varepsilon^d \mathfrak{f}_0^2 \sum_{i=1}^{K_\varepsilon} \Big(\Big\langle u(\varepsilon \boldsymbol{x}') \Big\rangle_{x' \in \varepsilon^{-1} Q_\varepsilon^i \cap F_0} - \Big\langle u(\varepsilon \boldsymbol{x}') \Big\rangle_{x' \in \varepsilon^{-1} Q_\varepsilon^i \cap F_1} \Big)^2 \tag{3.43}$$

$$\leq \varepsilon^d \mathfrak{f}_0 \mathfrak{f}_1^{-1} C_P(Q) \sum_{i=1}^{K_\varepsilon} \int\limits_{\varepsilon^{-1} Q_\varepsilon^i} \Big| \nabla_{x'} u(\varepsilon \boldsymbol{x}') \Big|^2 \mathrm{d}\boldsymbol{x}' = \mathfrak{f}_0 \mathfrak{f}_1^{-1} C_P(Q) \varepsilon^2 \sum_{i=1}^{K_\varepsilon} \int\limits_{Q_\varepsilon^i} \Big| \nabla u(\boldsymbol{x}) \Big|^2 \mathrm{d}\boldsymbol{x} \tag{3.44}$$

$$= \mathfrak{f}_0 \mathfrak{f}_1^{-1} C_P(Q) \varepsilon^2 \int\limits_{\mathbf{T}} \Big| \nabla u(\boldsymbol{x}) \Big|^2 \mathrm{d}\boldsymbol{x} = \mathfrak{f}_0 \mathfrak{f}_1^{-1} C_P(Q) \varepsilon^2 \|\nabla u\|_{L^2(\mathbf{T})}^2, \tag{3.45}$$

where $\mathfrak{f}_0 = |F_0 \cap Q|$ and $\mathfrak{f}_1 = |F_1 \cap Q|$. In formula (3.41) we split the original expression into a sum of K_ε integrals over cells of size ε. To get (3.42) we use the fact that every integration in (3.41) is performed on a constant function, and then we make use of the property (3.29) of the operation $\langle \cdot \rangle$, where $D_0 = Q_\varepsilon^i \cap F_0^\varepsilon$ and $D_1 = Q_\varepsilon^i \cap F_1^\varepsilon$. After that, by rescaling the variable of integration we obtain (3.43), which we estimate using inequality (3.40) with $\Omega = Q$. Finally, in (3.44) we rescale the variable of integration back and get (3.45) by additivity of the integral.[5] Hence, it is proved that

$$\|\bar{u}-\tilde{u}\|_{L^2(\mathbf{T})} \leq C_2 \varepsilon \|\nabla u\|_{L^2(\mathbf{T})}, \tag{3.46}$$

where $C_2 = \sqrt{\mathfrak{f}_0 \mathfrak{f}_1^{-1} C_P(Q)}$.

Note that

$$\|\bar{u}-\tilde{u}\|_{L^2(\mathbf{T})}^2 = \sum_{i=1}^{K_\varepsilon} \int\limits_{Q_\varepsilon^i} \Big| \langle u \rangle_{Q_\varepsilon^i} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big|^2 \mathrm{d}\boldsymbol{x} = \varepsilon^d \sum_{i=1}^{K_\varepsilon} \Big| \langle u \rangle_{Q_\varepsilon^i} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big|^2,$$

and therefore it follows from the inequality (3.46) that

$$\sum_{i=1}^{K_\varepsilon} \Big| \langle u \rangle_{Q_\varepsilon^i} - \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \Big|^2 \leq C_2^2 \varepsilon^{2-d} \|\nabla u\|_{L^2(\mathbf{T})}^2. \tag{3.47}$$

Finally, we estimate the third term in the right-hand side of (3.31). To this end,

[5] Note that in spite of the assumptions $\mathfrak{f}_0 \neq 0$, $\mathfrak{f}_1 \neq 0$ in the Lemma above, we can drop the first of them on the course of getting the estimate (3.41)–(3.45). Clearly, if $\mathfrak{f}_0 = 0$ then $\|\bar{u}-\tilde{u}\|_{L^2(\mathbf{T})} = 0$.

notice that due to the fact that

$$\sum_{i=1}^{K_\varepsilon} \langle u \rangle_{Q_\varepsilon^i} = \varepsilon^{-d} \langle u \rangle_{\mathbf{T}} = 0,$$

the following formulas hold

$$\left| \sum_{i=1}^{K_\varepsilon} \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \right| = \left| \sum_{i=1}^{K_\varepsilon} \Big(\langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} - \langle u \rangle_{Q_\varepsilon^i} \Big) \right|$$

$$\leq \sum_{i=1}^{K_\varepsilon} \left| \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} - \langle u \rangle_{Q_\varepsilon^i} \right| \leq \left(K_\varepsilon \sum_{i=1}^{K_\varepsilon} \left| \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} - \langle u \rangle_{Q_\varepsilon^i} \right|^2 \right)^{\frac{1}{2}}. \tag{3.48}$$

Using inequality (3.47) we can continue (3.48) to get

$$\left| \sum_{i=1}^{K_\varepsilon} \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} \right| \leq C_2 \Big(K_\varepsilon \varepsilon^{2-d} \Big)^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbf{T})}. \tag{3.49}$$

We claim that there exists a positive constant $\hat{c}$ such that the following inequality holds

$$\sum_{i,j=1}^{K_\varepsilon} \left| \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} - \langle u \rangle_{Q_\varepsilon^j \cap F_1^\varepsilon} \right|^2 \leq \hat{c} \varepsilon^{-2d} \|\nabla u\|^2_{L^2(\mathbf{T} \cap F_1^\varepsilon)}. \tag{3.50}$$

To show this, we use the following particular case of the above Lemma.

Let Q and Q' be two cells that share a common side. Then there exists a positive constant $\tilde{c}$ depending only on F_1 such that for any function $v \in H^1\Big((Q \cup Q') \cap F_1\Big)$ the following inequality holds

$$\left| \langle v \rangle_{Q \cap F_1} - \langle v \rangle_{Q' \cap F_1} \right| \leq \tilde{c} \|\nabla v\|_{L^2((Q \cup Q') \cap F_1)}. \tag{3.51}$$

If Q_ε^i and $Q^j\varepsilon$ share a common side then by use of (3.51) and rescaling we get

$$\left| \langle u \rangle_{Q_\varepsilon^i \cap F_1^\varepsilon} - \langle u \rangle_{Q_\varepsilon^j \cap F_1^\varepsilon} \right|^2 \leq \tilde{c} \varepsilon^{2-d} \Big(\|\nabla u\|^2_{L^2(Q_\varepsilon^i \cap F_1^\varepsilon)} + \|\nabla u\|^2_{L^2(Q^i\varepsilon \cap F_1^\varepsilon)} \Big). \tag{3.52}$$

Let $(m_k^1, m_k^2, ..., m_k^d)$ be the d-tuple of coordinates of the centre of the cell Q_ε^k, $k = 1, ..., K_\varepsilon$. We fix two arbitrary cells Q_ε^i and Q_ε^j and construct a "path" across consecutive cells in the cube $\mathbf{T}$, such that Q_ε^i and Q_ε^j are its "endpoints", in the following way. First, we move along the segment $x_k(t) = m_i^1 + t(m_j^1 - m_i^1)$, $t \in [0, 1]$ so that only the first coordinate of cells is changing, while the others are fixed and equal to those of the starting endpoint. Upon reaching the cell $(m_j^1, m_i^2, ..., m_i^d)$, we change the direction of the path so that at the second leg of the path only the second coordinate is changing. Upon reaching the cell $(m_j^1, m_j^2, ..., m_i^d)$, we turn again, and so on, till we reach the cell $(m_j^1, m_j^2, ..., m_j^d)$. Let us now number the cells that are members of the constructed path in order of passing them, from 1 to M_{ij}, where M_{ij} is the total number of cells involved in the path, so that the cell Q_ε^i has number 1 and the cell Q_ε^j has number M_{ij}. Obviously, $M_{ij} \le 2dN$, where $N = \varepsilon^{-1}T$. Let us introduce a temporary notation Q_r for the cell with number r in the path, $r = 1, ..., M_{ij}$. It follows from inequality (3.52) that

$$\left|\langle u\rangle_{Q_\varepsilon^i\cap F_1^\varepsilon} - \langle u\rangle_{Q_\varepsilon^j\cap F_1^\varepsilon}\right|^2 = \left|\sum_{r=1}^{M_{ij}-1}\left(\langle u\rangle_{Q_r} - \langle u\rangle_{Q_{r+1}}\right)\right|^2 \le (M_{ij}-1)\sum_{r=1}^{M_{ij}-1}\left|\langle u\rangle_{Q_r} - \langle u\rangle_{Q_{r+1}}\right|^2$$

$$\le (M_{ij}-1)\tilde{c}\varepsilon^{2-d}\sum_{r=1}^{M_{ij}-1}\left(\|\nabla u\|^2_{L^2(Q_r)} + \|\nabla u\|^2_{L^2(Q_{r+1})}\right) \le 4dN\tilde{c}\varepsilon^{2-d}\sum_{r=1}^{M_{ij}-1}\|\nabla u\|^2_{L^2(Q_r)}. \tag{3.53}$$

We perform the above procedure for any pair (i,j) of endpoints. It is not difficult to see that the total amount of times that any particular cell is come across during this process does not exceed $2(2N)^{d+1}$. Therefore, summing (3.53) over all possible i, j we get

$$\sum_{i,j=1}^{K_\varepsilon}\left|\langle u\rangle_{Q_\varepsilon^i\cap F_1^\varepsilon} - \langle u\rangle_{Q_\varepsilon^j\cap F_1^\varepsilon}\right|^2$$

$$\le 2^{d+5}dN^{d+2}\tilde{c}\varepsilon^{2-d}\|\nabla u\|^2_{L^2(\mathbf{T})} = 2^{d+5}dT^{d+2}\tilde{c}\varepsilon^{-2d}\|\nabla u\|^2_{L^2(\mathbf{T})}.$$

Hence it is proved that (3.50) holds with $\hat{c} = 2^{d+5}dT^{d+2}\tilde{c}$.

Using inequalities (3.49) and (3.50) we can now complete estimation of the third term in the right-hand side of (3.31)

$$\|\tilde{u}\|^2_{L^2(\mathbf{T})} = \varepsilon^d\sum_i\langle u\rangle^2_{Q_\varepsilon^i\cap F_1^\varepsilon} = \varepsilon^d(2K_\varepsilon)^{-1}\Bigg(\sum_{i,j}\left|\langle u\rangle_{Q_\varepsilon^i\cap F_1^\varepsilon} - \langle u\rangle_{Q_\varepsilon^j\cap F_1^\varepsilon}\right|^2$$

$$+2\Big(\sum_i\langle u\rangle_{Q_\varepsilon^i\cap F_1^\varepsilon}\Big)^2\Bigg) \le \varepsilon^d(2K_\varepsilon)^{-1}\hat{c}\varepsilon^{-2d}\|\nabla u\|^2_{L^2(\mathbf{T}\cap F_1^\varepsilon)}$$

$$+\varepsilon^d(2K_\varepsilon)^{-1}C_2^2 2K_\varepsilon\varepsilon^{2-d}\|\nabla u\|^2_{L^2(\mathbf{T})} = \tilde{C}^2\|\nabla u\|^2_{L^2(\mathbf{T}\cap F_1^\varepsilon)} + C_2^2\varepsilon^2\|\nabla u\|^2_{L^2(\mathbf{T})},$$

where $\tilde{C} = 4T\sqrt{d\tilde{c}}$, and hence

$$\|\tilde{u}\|_{L^2(\mathbf{T})} \le \tilde{C}\|\nabla u\|_{L^2(\mathbf{T}\cap F_1^\varepsilon)} + C_2\varepsilon\|\nabla u\|^2_{L^2(\mathbf{T})}. \tag{3.54}$$

Finally, adding up estimates (3.32), (3.46) and (3.54) we get the required inequality (3.27) with $C = \max\{C_1 + 2C_2, \tilde{C}\}$. $\square$

3.3 Non-local nature of the homogenised system (3.25)

3.3.1 Smoothness of the solution pair $(u^{(1)}, w)$

Now we intend to show how non-locality arises from consideration of the homogenised system (3.25). Before doing that let us make one more remark. Note, that the problem (3.25)–(3.26) is understood in the weak sense, *i.e.* in the sense of the identity (3.21). If the right-hand side $f(\boldsymbol{x})$ of the system belongs to the space $L^2(\mathbf{T})$ as it has so far, we can *a priori* claim that the solution $(u^{(1)}, w)$ belongs to the space $H^1_{per}(\mathbf{T}) \times L^2\Big(\mathbf{T}, H^1_{per}(Q)\Big)$. However, if the domain $\tilde{F}_0 \cap Q_2$ and the right-hand side of the original equation are smooth, namely, from now on we assume that $\partial(\tilde{F}_0 \cap Q_2) \in C^2$ and $f \in C^\infty_{per}(\mathbf{T})$, then the functions $u^{(1)}$ and w, forming the solution of the homogenised problem, are smooth, too. This will follow as a by-product in the following considerations.

Let us denote by $G = G(y_1, y_2, y_1', y_2', x_3)$ the Green's function of the operator $-\partial^2_{y_1} - \partial^2_{y_2} - \partial^2_{x_3}$ in the cylinder $(\tilde{F}_0 \cap Q_2) \times [-T, T]$ with periodic boundary condition on the surfaces $x_3 = \pm T$. The above Green's function exists, see *e.g.* Wermer [55]. From the second equation of the system (3.25) where $\lambda = 0$, we get

$$w(\boldsymbol{x}, \tilde{\boldsymbol{y}}) = \mathfrak{f}_0 \int_{-T}^{T} \Big\langle G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3 - x_3') \Big\rangle_{\tilde{\boldsymbol{y}}'} \Big(f(\tilde{\boldsymbol{x}}', x_3') + u^{(1)}_{,x_3' x_3'}(\tilde{\boldsymbol{x}}, x_3') \Big) \mathrm{d}x_3', \tag{3.55}$$

where $\Big\langle G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3 - x_3') \Big\rangle_{\tilde{\boldsymbol{y}}'}$ is the average of the Green's function with respect to the vector $\tilde{\boldsymbol{y}}' \in \tilde{F}_0 \cap Q_2$. We also write (3.55) in a concise form as follows

$$w = \mathfrak{f}_0 \langle G \rangle_{\tilde{y}'} \overset{x_3}{*} \Big(f + u^{(1)}_{,x_3 x_3} \Big), \tag{3.56}$$

where $\overset{x_3}{*}$ stands for convolution with respect to the variable x_3 only.

Substituting expression (3.56) for the function w into the first equation of the system (3.25) we obtain the following equation

$$-\mathrm{div}(A^{hom} \nabla u^{(1)}) - \mathfrak{f}_0^2 \langle G \rangle_{\tilde{y}', \tilde{y}} \overset{x_3}{*} \frac{\partial^4 u^{(1)}}{\partial x_3^4} = f + \mathfrak{f}_0^2 \langle G \rangle_{\tilde{y}', \tilde{y}} \overset{x_3}{*} \frac{\partial^2 f}{\partial x_3^2}. \tag{3.57}$$

To prove that a periodic solution to the equation (3.57) is infinitely smooth we use some standard techniques of Fourier analysis. Henceforth in this subsection we assume, without loss of generality, that $T = \pi$.

To this end observe that if we denote

$$F := f + \mathfrak{f}_0^2 \langle G \rangle_{\tilde{y}', \tilde{y}} \overset{x_3}{*} \frac{\partial^2 f}{\partial x_3^2}$$

then, formally, Fourier coefficients $\hat{u}_1(\boldsymbol{m})$ and $\hat{F}(\boldsymbol{m})$, $\boldsymbol{m} \in \mathbf{Z}^3$ of the periodic functions

$u^{(1)}(\boldsymbol{x})$ and $F(\boldsymbol{x})$ are related by the following formula

$$\left(\tilde{h}_{pr} m_p m_r + m_3^2 - \mathfrak{f}_0^2 m_3^4 \left\langle \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) \right\rangle_{\tilde{y}} \right) \hat{u}_1(\boldsymbol{m}) = \hat{F}(\boldsymbol{m}). \tag{3.58}$$

Here

$$\tilde{h}_{ij} := \int\limits_{\tilde{F}_1 \cap Q_2} \Big(\delta_{ij} + (N_i)_{,j}(\tilde{\boldsymbol{y}})\Big) \mathrm{d}\tilde{\boldsymbol{y}}, \quad i, j = 1, 2 \tag{3.59}$$

are elements of the matrix A_{2D}^{hom} (see (3.7)), and $\hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3)$ are the Fourier coefficients of the function $\mathcal{K}(\tilde{\boldsymbol{y}}, x_3) := \left\langle G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3) \right\rangle_{\tilde{y}'}$. Note that the Green's function G is the x_3-periodic solution to the following boundary value problem

$$-\frac{\partial^2 G}{\partial y_1^2} - \frac{\partial^2 G}{\partial y_2^2} - \frac{\partial^2 G}{\partial x_3^2} = \delta(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}', x_3), \quad (\tilde{\boldsymbol{y}}, x_3), (\tilde{\boldsymbol{y}}', x_3) \in (\tilde{F}_0 \cap Q_2) \times [-T, T], \tag{3.60}$$

$$G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3)\Big|_{\tilde{y} \in \partial \tilde{F}_0 \cap Q_2} = 0. \tag{3.61}$$

Taking the average with respect to $\tilde{\boldsymbol{y}}' \in \tilde{F}_0 \cap Q_2$ in (3.60)–(3.60) we arrive at a boundary value problem for the function $\mathcal{K}(\tilde{\boldsymbol{y}})$ as follows

$$-\frac{\partial^2 \mathcal{K}}{\partial y_1^2} - \frac{\partial^2 \mathcal{K}}{\partial y_2^2} - \frac{\partial^2 \mathcal{K}}{\partial x_3^2} = \mathfrak{f}_0^{-1} \delta(x_3), \quad (\tilde{\boldsymbol{y}}, x_3) \in (\tilde{F}_0 \cap Q_2) \times [-T, T],$$

$$\mathcal{K}(\tilde{\boldsymbol{y}}, x_3)\Big|_{\tilde{y} \in \partial \tilde{F}_0 \cap Q_2} = 0.$$

Hence the following boundary value problem for the Fourier coefficients $\hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3)$ holds

$$-\frac{\partial^2 \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3)}{\partial y_1^2} - \frac{\partial^2 \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3)}{\partial y_2^2} + m_3^2 \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) = \mathfrak{f}_0^{-1}, \quad \tilde{\boldsymbol{y}} \in \tilde{F}_0 \cap Q_2, \tag{3.62}$$

$$\hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3)\Big|_{\tilde{y} \in \partial \tilde{F}_0 \cap Q_2} = 0. \tag{3.63}$$

We treat $m_3 \in \mathbf{Z}$ in (3.62)–(3.63) as a parameter. It is easy to see that

$$\hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) = \mathfrak{f}_0^{-1} \int\limits_{\tilde{F}_0 \cap Q_2} \mathcal{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}', \tag{3.64}$$

where $\mathcal{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3)$ is the Green's function of the operator $-\partial_{y_1}^2 - \partial_{y_2}^2 + m_3^2$ in the domain $\tilde{F}_0 \cap Q_2$. The formula (3.64) implies that

$$\left\langle \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) \right\rangle_{\tilde{y}} = \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{\tilde{F}_0 \cap Q_2} \mathcal{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}}. \tag{3.65}$$

The Green's function $\mathcal{G}$ can be split into two parts, the fundamental solution $\mathcal{E}$ of the operator $-\partial_{y_1}^2 - \partial_{y_2}^2 + m_3^2$ in $\mathbf{R}^2$ and the "reflected" part $\tilde{\mathcal{G}}$ as follows

$$\mathcal{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) = \mathcal{E}(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}', m_3) + \tilde{\mathcal{G}}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3). \tag{3.66}$$

The fundamental solution $\mathcal{E}$ is well known, namely,

$$\mathcal{E}(\boldsymbol{z}, m_3) = \frac{1}{2\pi} K_0\Big(|m_3||\boldsymbol{z}|\Big), \quad \boldsymbol{z} \in \mathbf{R}^2, \tag{3.67}$$

where K_0 is a modified Bessel function of the third kind.

Due to the formula (3.66), the value $\Big\langle \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) \Big\rangle_{\tilde{y}}$ splits into two corresponding parts as follows

$$\Big\langle \hat{\mathcal{K}}(\tilde{\boldsymbol{y}}, m_3) \Big\rangle_{\tilde{y}} = \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{\tilde{F}_0 \cap Q_2} \mathcal{E}(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}} + \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{\tilde{F}_0 \cap Q_2} \tilde{\mathcal{G}}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}}. \tag{3.68}$$

It is relatively easy to estimate the first term in the right-hand side of (3.68). To this end notice that using the formula (3.67) we get

$$\begin{aligned} \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{\tilde{F}_0 \cap Q_2} \mathcal{E}(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}} &\leq \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{R^2} \mathcal{E}(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}} \\ &= \mathfrak{f}_0^{-1} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{R^2} \mathcal{E}(\boldsymbol{z}, m_3) \mathrm{d}\tilde{\boldsymbol{z}} \mathrm{d}\tilde{\boldsymbol{y}}' \\ &= \frac{\mathfrak{f}_0^{-1}}{2\pi} \int\limits_{R^2} K_0\Big(|m_3||\boldsymbol{z}|\Big) \mathrm{d}\tilde{\boldsymbol{z}} = \mathfrak{f}_0^{-1} m_3^{-2} \int\limits_0^\infty K_0(\zeta)\zeta \mathrm{d}\zeta = \mathfrak{f}_0^{-1} m_3^{-2}. \end{aligned} \tag{3.69}$$

The last equality follows from the following formula

$$\int\limits_0^\infty K_0(\zeta)\zeta \mathrm{d}\zeta = 1, \tag{3.70}$$

which can be found *e.g.* in Gradshteyn & Ryzhik [25].

We proceed with estimating the second integral in the right-hand side of (3.68). To this end, we introduce the set $\tilde{F}_0^{m_3} = \Big\{ \tilde{\boldsymbol{y}} \in \tilde{F}_0 \cap Q_2 : \operatorname{dist}(\tilde{\boldsymbol{y}}, \partial \tilde{F}_0 \cap Q_2) < |m_3|^{-1} \Big\}$, and consider the system of coordinates in $\tilde{F}_0^{m_3}$ associated with the boundary and the inner normal to the boundary. More precisely, in this coordinate system each point $\tilde{\boldsymbol{y}}_0 \in \tilde{F}_0^{m_3}$ has coordinates $(n, s) = \Big(n(\tilde{\boldsymbol{y}}), s(\tilde{\boldsymbol{y}})\Big)$, where $n(\tilde{\boldsymbol{y}})$ is the distance of the point to the boundary, and $s(\tilde{\boldsymbol{y}})$ is the arc length measured along the boundary from some fixed point $\tilde{\boldsymbol{y}}_0 \in \partial \tilde{F}_0 \cap Q_2$ to the orthogonal projection of the point $\tilde{\boldsymbol{y}}_0$ on the boundary.

The function $\tilde{\mathcal{G}}$ is non-positive as follows from the maximum principle. Therefore,

$$I := \mathfrak{f}_0^{-2} \int\limits_{\tilde{F}_0 \cap Q_2} \int\limits_{\tilde{F}_0 \cap Q_2} \tilde{\mathcal{G}}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}} \leq \mathfrak{f}_0^{-2} \int\limits_{\Omega^{m_3}} \tilde{\mathcal{G}}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m_3) \mathrm{d}\tilde{\boldsymbol{y}}' \mathrm{d}\tilde{\boldsymbol{y}}, \tag{3.71}$$

where $\Omega^{m_3} = \left\{ (\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}') : \tilde{\boldsymbol{y}} \in \tilde{F}_0^{m_3}, \tilde{\boldsymbol{y}} \in \tilde{F}_0^{m_3}, \left| s(\tilde{\boldsymbol{y}}) - s(\tilde{\boldsymbol{y}}') \right| < |m_3|^{-1} \right\}$.

Making the following changes of variables in the last integral $\tilde{\boldsymbol{y}} \to \Big(n(\tilde{\boldsymbol{y}}), s(\tilde{\boldsymbol{y}}) \Big)$, $\tilde{\boldsymbol{y}}' \to \Big(n'(\tilde{\boldsymbol{y}}'), s'(\tilde{\boldsymbol{y}}') \Big)$, we arrive at the following inequality

$$I \leq \mathfrak{f}_0^{-2} \int\limits_0^L \mathrm{d}s \int\limits_0^{|m_3|^{-1}} \mathrm{d}n \int\limits_{s-|m_3|^{-1}}^{s+|m_3|^{-1}} \mathrm{d}s' \int\limits_0^{|m_3|^{-1}} \mathrm{d}n' \tilde{\mathcal{G}}(s, n, s', n') \Big| J(s,n) \Big|^{-1} \Big| J(s', n') \Big|^{-1}, \tag{3.72}$$

where L is the length of $\partial \tilde{F}_0 \cap Q_2$ and $J(s,n)$ and $J(s', n')$ are the corresponding Jacobians of the changes of variables. It is easy to verify that if the boundary of the domain $\tilde{F}_0 \cap Q_2$ is the graph of a function $z_2 = g(z_1)$ in the Cartesian system of coordinates $\tilde{\boldsymbol{z}} = (z_1, z_2)$ generated by the tangent to $\partial \tilde{F}_0 \cap Q_2$ at the point $\tilde{\boldsymbol{y}} \in \partial \tilde{F}_0 \cap Q_2$, then the Jacobi matrix $\mathbf{J}(s, n)$ is given by the following formula

$$\mathbf{J}\Big(s(\tilde{\boldsymbol{y}}), n(\tilde{\boldsymbol{y}}) \Big) = \begin{pmatrix} n(\tilde{\boldsymbol{z}}) \Big| g''(z_1) \Big| + 1 & 0 \\ 0 & 1 \end{pmatrix} R, \tag{3.73}$$

where R is a rotation matrix. Using the assumption that the boundary is twice continuously differentiable we conclude that in the domain $0 \leq s \leq L, 0 \leq n \leq |m_3|^{-1}$

$$\Big| J(s,n) \Big| = 1 + O\Big(|m_3|^{-1} \Big).$$

The function $\tilde{\mathcal{G}}(s, n, s', n')$ is the result of "reflection" of the fundamental solution of the operator $-\partial_{y_1}^2 - \partial_{y_2}^2 + m_3^2$ about the boundary of the domain $\tilde{F}_0 \cap Q_2$, that is

$$\tilde{\mathcal{G}}(s, n, s', n', m_3) = -\mathcal{E}\Big(z_1(s,n) - z_1(s', -n'), z_2(s,n) - z_2(s', n'), m_3 \Big). \tag{3.74}$$

Note that the Taylor formula and the expression for the Jacobi matrix $\mathbf{J}(s,n) R^{-1}$ of the transformation $\tilde{z} = \tilde{z}(s,n)$ (see (3.73)) imply that

$$\begin{aligned} z_1(s,n) - z_1(s', -n') &= y_1(s', 0) + (y_1)_{,s}(s', 0)(s - s') + (y_1)_{,n}(s', 0) n \\ +O\Big((s-s')^2 + n^2 \Big) &- y_1(s', 0) + (y_1)_{,n}(s', 0) n' + O\Big((s-s')^2 + (n')^2 \Big) \\ &= s - s' + O\Big((s-s')^2 + n^2 + (n')^2 + n|s - s'| \Big) \end{aligned} \tag{3.75}$$

and

$$z_2(s,n) - z_2(s', n') = y_2(s', 0) + (y_2)_{,s}(s', 0)(s - s') + (y_2)_{,n}(s', 0) n$$

$$+O\Big((s-s')^2+n^2\Big)-y_2(s',0)+(y_2)_{,n}(s',0)n'+O\Big((s-s')^2+(n')^2\Big)$$

$$=n+n'+O\Big((s-s')^2+n^2+(n')^2+n|s-s'|\Big). \tag{3.76}$$

Using formulas (3.74), (3.75) and (3.76) we conclude that

$$\tilde{\mathcal{G}}(s,n,s',n',m_3)=-\mathcal{E}(s-s',n+n',m_3)\Bigg(1+O\Big((s-s')^2+n^2+(n')^2+n|s-s'|\Big)\Bigg). \tag{3.77}$$

We make the change of variables in the integral (3.72) as follows $\tilde{n}=|m_3|n$, $\tilde{n}'=|m_3|n'$, $\tilde{s}'=|m_3|(s-s')$ and use formula (3.77) to obtain

$$I\le -CL\mathfrak{f}_0^{-2}|m_3|^{-3}\int\limits_0^1 \mathrm{d}\tilde{n}\int\limits_{-1}^1 \mathrm{d}\tilde{s}'\int\limits_0^1 \mathrm{d}\tilde{n}'\,\mathcal{E}\Big(|m_3|^{-1}\tilde{s}',|m_3|^{-1}(\tilde{n}+\tilde{n}'),m_3\Big) \tag{3.78}$$

for some positive constant C, depending only on the maximal curvature of the boundary of the domain $\tilde{F}_0\cap Q_2$.

Taking into account expression (3.67) for the fundamental solution $\mathcal{E}$ we arrive at the following inequality

$$I\le -CL\mathfrak{f}_0^{-2}(2\pi)^{-1}|m_3|^{-3}\int\limits_0^1 \mathrm{d}\tilde{n}\int\limits_{-1}^1 \mathrm{d}\tilde{s}'\int\limits_0^1 \mathrm{d}\tilde{n}'\,K_0\Big(\sqrt{(\tilde{s}')^2+(\tilde{n}+\tilde{n}')^2}\Big) \tag{3.79}$$

Finally, from (3.79) and (3.70) we get

$$I\le -C\mathfrak{f}_0^{-2}(2\pi)^{-1}|m_3|^{-3}\int\limits_{R^2} K_0(|\boldsymbol{z}|)\mathrm{d}\boldsymbol{z}$$

$$=-CL\mathfrak{f}_0^{-2}|m_3|^{-3}\int\limits_0^\infty K_0(\zeta)\zeta\mathrm{d}\zeta=-CL\mathfrak{f}_0^{-2}|m_3|^{-3}. \tag{3.80}$$

Using (3.68), (3.69) and (3.80) we conclude that

$$\Big\langle\hat{\mathcal{K}}(\tilde{\boldsymbol{y}},m_3)\Big\rangle_{\tilde{\boldsymbol{y}}}\le \mathfrak{f}_0^{-1}m_3^{-2}-CL\mathfrak{f}_0^{-2}|m_3|^{-3}.$$

Recalling the equation (3.58) for the Fourier coefficients of the function $u^{(1)}(\boldsymbol{x})$, we get

$$\hat{u}_1(\boldsymbol{m})=\Big(V(\boldsymbol{m})\Big)^{-1}\hat{F}(\boldsymbol{m}), \tag{3.81}$$

where

$$V(\boldsymbol{m}):=\tilde{h}_{pr}m_pm_r+m_3^2-\mathfrak{f}_0^2m_3^4\Big\langle\hat{\mathcal{K}}(\tilde{\boldsymbol{y}},m_3)\Big\rangle_{\tilde{\boldsymbol{y}}}\ge \tilde{h}_{pr}m_pm_r+\mathfrak{f}_1m_3^2+CL|m_3|. \tag{3.82}$$

Due to the fact that the function $F(\boldsymbol{x})$ is infinitely smooth, its Fourier coefficients decay when $|\boldsymbol{m}| \to \infty$, faster than any power of $|\boldsymbol{m}|$. Therefore, in view of the formula (3.81) and the estimate (3.82), the Fourier coefficients of the function $u^{(1)}(\boldsymbol{x})$ decay faster than any power of $|\boldsymbol{m}|$ as well, which implies smoothness of the function $u^{(1)}$. Hence, due to the formula (3.55), the function $w(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ is infinitely smooth, too. Using uniqueness of the weak solution to the homogenised problem (3.25)–(3.26), we conclude that this weak solution is infinitely smooth.

3.3.2 Asymptotic expansion of the solution and its justification.

There is an alternative way to perform homogenisation of the equation (3.2) and it is employing the method of double-scale asymptotic expansion. It is plausible to expect that if we consider the ansatz we already used in Chapter 1 and Chapter 2, we can recover both the leading term of the asymptotics of the solution of (3.2) and the homogenised system of equations (3.25). Moreover, as we show below, the full asymptotic expansion of the solution of (3.2) can be constructed and its remainder can be bounded. Having this objective in mind, we consider the following general formal asymptotic expansion separating slow and fast variables (*cf.* (1.5), (2.9))

$$u^{\varepsilon}(\boldsymbol{x}) \sim \sum_{l=0}^{\infty} \varepsilon^{l} u_l(\boldsymbol{x}, \boldsymbol{x}/\varepsilon), \tag{3.83}$$

where the functions $u_l(\boldsymbol{x}, \boldsymbol{y})$ are sought to be $\mathbf{T}$-periodic in $\boldsymbol{x}$ and Q-periodic in $\boldsymbol{y}$.

Henceforth in this subsection we assume that $\lambda = 0$ in the equation (3.2). This considerably simplifies exposition of the work and, as we have seen before, does not lead to loss of generality in a certain sense. Hence, we consider the equation

$$-\Big(A^{\varepsilon}_{ij}(\boldsymbol{x}/\varepsilon) u_{,j}\Big)_{,i} = f(\boldsymbol{x}), \tag{3.84}$$

where the matrix $\Big(A_{ij}(\boldsymbol{y})\Big)$ is given by the formula (3.1).

In the formulas of this subsection, we denote the restrictions of $u_l(\boldsymbol{x}, \boldsymbol{y})$ to the sets $y \in F_0 \cap Q$ and $y \in F_1 \cap Q$ by u_l^- and u_l^+, respectively. Note, that not only must the series (3.83) formally satisfy the equation (3.2), but it also ought to satisfy the conditions on the interface between the soft and hard phases $F_0 \cap Q$ and $F_1 \cap Q$, which are given next

$$\Big[u^{\varepsilon}(\boldsymbol{x})\Big]_{\partial F_0 \cap Q} = 0, \qquad \Big[A^{\varepsilon}_{ij}(\boldsymbol{x}/\varepsilon) u^{\varepsilon}_{,i}(\boldsymbol{x}) n_j(\boldsymbol{x}/\varepsilon)\Big]_{\partial F_0 \cap Q} = 0, \tag{3.85}$$

where $[\,\cdot\,]_{\partial F_0 \cap Q}$ stands for the jump of the function in the brackets across the interface $\partial F_0 \cap Q$ between the phases, and $\boldsymbol{n}(\boldsymbol{y}) = \Big(n_1(\boldsymbol{y}), n_2(\boldsymbol{y}), n_3(\boldsymbol{y})\Big)^{\top}$ is the unit normal vector to $\partial F_0 \cap Q$ at the point $\boldsymbol{y}$ directed towards interior of $F_0 \cap Q$. Note in passing that the normal $\boldsymbol{n}(\boldsymbol{y})$ depends only on $\tilde{\boldsymbol{y}}$ and its third component is zero, *i.e.* $\boldsymbol{n}(\tilde{\boldsymbol{y}}) = \Big(n_1(\tilde{\boldsymbol{y}}), n_2(\tilde{\boldsymbol{y}}), 0\Big)^{\top}$. Hereafter we denote $\tilde{\boldsymbol{n}}(\tilde{\boldsymbol{y}}) := \Big(n_1(\tilde{\boldsymbol{y}}), n_2(\tilde{\boldsymbol{y}})\Big)^{\top}$.

Bearing these remarks in mind, we substitute the series (3.83) in the equation (3.84) and interface jump conditions (3.85), perform necessary formal differentiations and gather the terms of the same order in ε together. Note that the first of the conditions (3.85) results simply in continuity of each of the functions $u_l(\boldsymbol{x}, \boldsymbol{y})$ with respect to $\boldsymbol{y}$ across $\partial F_0 \cap Q$. The role of the second condition is going to be clear from further explanations and henceforth it will be referred to as the condition on the interface (3.85). As a result of the outlined procedure, we obtain a sequence of recurrence relations on the functions u_l as described below.

Gathering the terms of order ε^{-2} together, we see that function u_0^- must satisfy the equation

$$(u_0^-)_{,y_3y_3} = 0.$$

This and the requirement of $[0,1]$-periodicity of the function u_0^- with respect to the variable y_3 imply the fact that u_0^- is independent of y_3. On the other hand, the equation for the function u_0^+ (arising from consideration of the term of order ε^{-2} in the hard phase) is the following

$$\Delta_y u_0^+ = 0. \tag{3.86}$$

The term of order ε^{-1} of the condition on the interface gives us a boundary condition for u_0^+ as follows

$$\left.\frac{\partial u_0^+}{\partial \boldsymbol{n}}\right|_{\partial F_0 \cap Q} = 0. \tag{3.87}$$

The equation (3.86) together with the condition (3.87) imply that the function u_0^+ does not depend on $\boldsymbol{y}$.

Next, the term of order ε^{-1} in the soft phase gives us

$$(u_1^-)_{,y_3y_3} = 0,$$

which implies that u_1^- is independent of y_3; and in the hard phase it gives

$$\Delta_y u_1^+ = 0,$$

which together with the condition

$$\left.\frac{\partial u_1^+}{\partial \boldsymbol{n}}\right|_{\partial F_0 \cap Q} = -\nabla u_0^+ \cdot \boldsymbol{n}$$

obtained by looking at the term of order 1 in the condition on the interface (3.85), results in the following formula for u_1:

$$u_1^+(\boldsymbol{x}, \boldsymbol{y}) = N_1(\tilde{\boldsymbol{y}})(u_0^+)_{,x_1} + N_2(\tilde{\boldsymbol{y}})(u_0^+)_{,x_2}. \tag{3.88}$$

Here, $N_1(\tilde{\boldsymbol{y}})$ and $N_2(\tilde{\boldsymbol{y}})$ are the solutions of the standard "unit-cell" problems on $\tilde{F}_1 \cap Q$

as follows

$$\Delta N_i = 0 \ \text{ in } \tilde{F}_1 \cap Q_2, \quad \frac{\partial N_i}{\partial \tilde{n}} = -n_i \ \text{ on } \partial \tilde{F}_0 \cap Q_2, \quad i = 1, 2. \tag{3.89}$$

Proceeding further, we consider the term of order 1 in the soft phase to get the following equation for the function u_2^- :

$$-(u_2^-)_{,y_3y_3} = f + (u_0^-)_{,y_1y_1} + (u_0^-)_{,y_2y_2} + (u_0^-)_{,x_3x_3}$$

The solvability condition for the last equation is that the average of the right-hand side with respect to y_3 be zero. It provides us with the equation on the function u_0^- as follows

$$-(u_0^-)_{,y_1y_1} - (u_0^-)_{,y_2y_2} - (u_0^-)_{,x_3x_3} = f. \tag{3.90}$$

The equation (3.90) is considered together with the following boundary condition

$$u_0^-|_{\partial F_0 \cap Q} = u_0^+|_{\partial F_0 \cap Q}, \tag{3.91}$$

which comes from the continuity condition for u^ε on the interface.

By grouping together the terms of order 1 in the hard phase, we arrive at the following equation for the function u_2^+ :

$$-\Delta_y u_2^+ = f + \Delta u_0^+ + 2\sum_{i=1}^{3} \frac{\partial^2 u_1^+}{\partial x_i \partial y_i}. \tag{3.92}$$

This equation should be considered in conjunction with the condition

$$\left.\frac{\partial u_2^+}{\partial \boldsymbol{n}}\right|_{\partial F_0 \cap Q} = \left.\Big(-\nabla_x u_1^+ \cdot \boldsymbol{n} + \nabla_y u_0^- \cdot \boldsymbol{n}\Big)\right|_{\partial F_0 \cap Q}, \tag{3.93}$$

coming from consideration of the terms of order ε in the condition on the interface (3.85).

The solvability condition for the problem (3.92)–(3.93) is that the sum of the integral of the right-hand side of the equation (3.92) over $F_1 \cap Q$ and the surface integral of the normal derivative of u_2^+ over $\partial F_0 \cap Q$ be zero. Hence, we must require that

$$\mathfrak{f}_1 f + \mathfrak{f}_1 \Delta u_0^+ + 2\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int\limits_{F_1 \cap Q} \frac{\partial u_1^+}{\partial y_i} \mathrm{d}\boldsymbol{y} - \int\limits_{\partial F_0 \cap Q} \nabla u_1^+ \cdot \boldsymbol{n} \mathrm{d}S(\boldsymbol{y}) + \int\limits_{\partial F_0 \cap Q} \nabla_y u_0^- \cdot \boldsymbol{n} \mathrm{d}S(\boldsymbol{y}) = 0. \tag{3.94}$$

Due to the fact that $F_1 \cap Q = (\tilde{F}_1 \cap Q_2) \times [0, 1]$ and $n_3(\boldsymbol{y}) = 0$, the condition (3.94) can be rewritten in the following way

$$\mathfrak{f}_1 f + \mathfrak{f}_1 \Delta u_0^+ + 2\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \int\limits_{\tilde{F}_1 \cap Q_2} \frac{\partial u_1^+}{\partial y_i} \mathrm{d}\tilde{\boldsymbol{y}} - \int\limits_{\partial \tilde{F}_0 \cap Q} \nabla_{\tilde{x}} u_1^+ \cdot \tilde{\boldsymbol{n}} ds(\tilde{\boldsymbol{y}}) + \int\limits_{\partial \tilde{F}_0 \cap Q_2} \nabla_{\tilde{y}} u_0^- \cdot \tilde{\boldsymbol{n}} ds(\tilde{\boldsymbol{y}}) = 0, \tag{3.95}$$

where $ds(\tilde{\boldsymbol{y}})$ denotes the arc length differential along the corresponding curve. Clearly,

$$\int\limits_{\partial \tilde{F}_0 \cap Q} \nabla_{\tilde{x}} u_1^+ \cdot \tilde{\boldsymbol{n}} ds(\tilde{\boldsymbol{y}}) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \int\limits_{\tilde{F}_1 \cap Q_2} \frac{\partial u_1^+}{\partial y_i} \mathrm{d}\tilde{\boldsymbol{y}}$$

and therefore the equality (3.95) can be rewritten once again as

$$\mathfrak{f}_1 f + \mathfrak{f}_1 \Delta u_0^+ + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \int\limits_{\tilde{F}_1 \cap Q_2} \frac{\partial u_1^+}{\partial y_i} \mathrm{d}\tilde{\boldsymbol{y}} + \int\limits_{\partial \tilde{F}_0 \cap Q_2} \nabla_{\tilde{y}} u_0^- \cdot \tilde{\boldsymbol{n}} ds(\tilde{\boldsymbol{y}}) = 0. \tag{3.96}$$

We substitute the expression for u_1^+ given by the formula (3.88) in the third term of (3.96) to get

$$\mathfrak{f}_1 \Delta u_0^+ + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \int\limits_{\tilde{F}_1 \cap Q_2} \frac{\partial u_1^+}{\partial y_i} \mathrm{d}\tilde{\boldsymbol{y}} = \sum_{i,j=1}^{2} \tilde{h}_{ij} \frac{\partial^2 u_0^+}{\partial x_i \partial x_j} \tag{3.97}$$

and integrate by parts in the fourth term of (3.96), which together with the equation (3.90) gives

$$\int\limits_{\partial \tilde{F}_0 \cap Q_2} \nabla_{\tilde{y}} u_0^- \cdot \tilde{\boldsymbol{n}} ds(\tilde{\boldsymbol{y}}) = - \int\limits_{\partial \tilde{F}_0 \cap Q_2} \Delta_{\tilde{y}} u_0^- \mathrm{d}\tilde{\boldsymbol{y}} = \int\limits_{\partial \tilde{F}_0 \cap Q_2} \Big((u_0^-)_{,x_3 x_3} + f \Big) \mathrm{d}\tilde{\boldsymbol{y}} = \langle u_0^- \rangle_{,x_3 x_3} + \mathfrak{f}_0 f. \tag{3.98}$$

Formulas (3.96), (3.97) and (3.98) altogether deliver the following equation for u_0^+ :

$$-\mathrm{div}_{\tilde{x}} \Big(A_{2D}^{hom} \nabla_{\tilde{x}} u_0^+ \Big) - \langle u_0^- \rangle_{,x_3 x_3} = f, \tag{3.99}$$

where the matrix A_{2D}^{hom} is given by the formula (3.7).

Note that if we introduce a function $\tilde{w} = \tilde{w}(\boldsymbol{x}, \tilde{\boldsymbol{y}})$ defined in $\tilde{\boldsymbol{y}} \in \tilde{F}_0 \cap Q_2$ by the formula $\tilde{w}(\boldsymbol{x}, \tilde{\boldsymbol{y}}) = u_0^-(\boldsymbol{x}, \tilde{\boldsymbol{y}}) - u_0^+(\boldsymbol{x})$ then it follows from (3.90) and (3.99) that the pair $(u_0^+, \tilde{w})$ satisfies the following coupled system of equations

$$\begin{cases} -\mathrm{div}\Big(A^{hom} \nabla u_0^+ \Big) - \dfrac{\partial^2 \langle \tilde{w} \rangle}{\partial x_3^2} = f, \quad \boldsymbol{x} \in \mathbf{T}, \\ -\dfrac{\partial^2 \tilde{w}}{\partial y_1^2} - \dfrac{\partial^2 \tilde{w}}{\partial y_2^2} - \dfrac{\partial^2 \tilde{w}}{\partial x_3^2} - \dfrac{\partial^2 u_0^+}{\partial x_3^2} = f, \quad \tilde{\boldsymbol{y}} \in \tilde{F}_0 \cap Q_2, \end{cases} \tag{3.100}$$

and as follows from (3.91), the function $\tilde{w}$ satisfies the following boundary condition

$$\tilde{w}(\boldsymbol{x}, \tilde{\boldsymbol{y}}) \Big|_{\tilde{y} \in \tilde{F}_0 \cap Q_2} = 0. \tag{3.101}$$

Obviously, the problem (3.100)–(3.101) coincides with the problem (3.25)–(3.26) that emerged before as a result of use of the method of two-scale convergence. In particular, the uniqueness of its solution implies that $u_0^+ \equiv u^{(1)}$ and $\tilde{w} \equiv w$.

Following this formal asymptotic procedure, we obtain a sequence of recurrence relations defining the functions $u_l^+, l = 1, 2, ...$ and $u_l^-, l = 1, 2, ...$ as follows

$$-\frac{\partial^2 u_{l+2}^-}{\partial y_3^2} = 2\sum_{i=1}^{2} \frac{\partial^2 u_{l-1}^-}{\partial x_i \partial y_i} + \sum_{i=1}^{2} \frac{\partial^2 u_l^-}{\partial y_i^2} + \frac{\partial^2 u_l^-}{\partial x_3^2} + 2\frac{\partial^2 u_{l+1}^-}{\partial x_3 \partial y_3} \quad \text{in } F_0 \cap Q, \tag{3.102}$$

$$(\nabla_x u_{l-1}^-, \boldsymbol{n}) + \frac{\partial u_l^-}{\partial \boldsymbol{n}} = (\nabla_x u_{l+1}^+, \boldsymbol{n}) + \frac{\partial u_{l+2}^-}{\partial \boldsymbol{n}} \quad \text{on } \partial F_0 \cap Q, \tag{3.103}$$

$$-\Delta_y u_{l+2}^+ = 2\sum_{i=1}^{3} \frac{\partial^2 u_{l+1}^+}{\partial x_i \partial y_i} + \Delta_x u_l^+ \quad \text{in } F_1 \cap Q. \tag{3.104}$$

Note that the functions $u_l(\boldsymbol{x}, \boldsymbol{y})$, $l = 1, 2, ...$ are smooth with respect to the pair of variables $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{T} \times Q$, which can be easily seen along the lines of the previous subsection.

The constructed formal asymptotic expansion can be rigorously substantiated as follows. We denote by $R^{(K)}(\boldsymbol{x}, \varepsilon)$ the remainder of the asymptotic series (3.83) after the term $u_K(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$, *i.e.*

$$R^{(K)}(\boldsymbol{x}, \varepsilon) := u^\varepsilon(\boldsymbol{x}) - \sum_{l=0}^{K} \varepsilon^l u_l(\boldsymbol{x}, \boldsymbol{x}/\varepsilon).$$

We show that there exist positive constants $C^{(K)}$ and $\tilde{C}^{(K)}$, depending only on K such that

$$\left\| R^{(K)}(\boldsymbol{x}, \varepsilon) \right\|_{L^2(\mathbf{T})} \leq C^{(K)} \varepsilon^{K+1} \tag{3.105}$$

and

$$\left\| \nabla R^{(K)}(\boldsymbol{x}, \varepsilon) \right\|_{L^2(\mathbf{T})} \leq \tilde{C}^{(K)} \varepsilon^{K}. \tag{3.106}$$

To see this, we substitute the sum $\mathcal{U}_K := \sum_{l=0}^{K} \varepsilon^l u_l(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ in the equation (3.84), perform corresponding formal manipulations and use recurrence relations (3.102)–(3.104) to get

$$-\left(A_{ij}^\varepsilon(\boldsymbol{x}/\varepsilon)(\mathcal{U}_K)_{,i} \right)_{,j} = f(\boldsymbol{x}) + \varepsilon^{K-1} \theta_K(\boldsymbol{x}, \varepsilon),$$

where $\left| \theta_K(\boldsymbol{x}, \varepsilon) \right| \leq C_K(f)$ and $C_K(f)$ is independent of ε, due to smoothness of the functions $u_l(\boldsymbol{x}, \boldsymbol{y})$ with respect to $\boldsymbol{x} \in \mathbf{T}$ and $\boldsymbol{y} \in Q$.

Therefore, the remainder $R^{(K)}$ satisfies an equation as follows

$$-\left(A_{ij}^\varepsilon(\boldsymbol{x}/\varepsilon) R_{,i}^{(K)} \right)_{,j} = \varepsilon^{K-1} \theta_K(\boldsymbol{x}, \varepsilon).$$

Multiplying both parts of the last equation by $R^{(K)}$ and integrating by parts we get

$$\int\limits_{\mathbf{T}} A_{ij}^{\varepsilon}(\boldsymbol{x}/\varepsilon) R_{,i}^{(K)} R_{,j}^{(K)} \mathrm{d}\boldsymbol{x} = \varepsilon^{K-1} \int\limits_{\mathbf{T}} \theta_K R^{(K)} \mathrm{d}\boldsymbol{x}.$$

Taking into account the definition of the matrix $\left(A_{ij}^{\varepsilon}(\boldsymbol{y})\right)$ we rewrite the last equation in the following way:

$$\int\limits_{\mathbf{T}\cap F_1^{\varepsilon}} R_{,i}^{(K)} R_{,i}^{(K)} \mathrm{d}\boldsymbol{x} + \varepsilon^2 \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} \left(\left(R_{,1}^{(K)}\right)^2 + \left(R_{,2}^{(K)}\right)^2 \right) \mathrm{d}\boldsymbol{x}$$

$$+ \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} \left(R_{,3}^{(K)}\right)^2 \mathrm{d}\boldsymbol{x} = \varepsilon^{K-1} \int\limits_{\mathbf{T}} \theta_K R^{(K)} \mathrm{d}\boldsymbol{x}. \tag{3.107}$$

From (3.107), using the Poincaré-type inequality (3.27) proved above we get

$$\left\| R^{(K)} \right\|_{L^2(\mathbf{T})}^2 \leq \varepsilon^{K-1} \left| \int\limits_{\mathbf{T}} \theta_K R^{(K)} \mathrm{d}\boldsymbol{x} \right|. \tag{3.108}$$

We apply the Hölder inequality to the right-hand side of the inequality (3.108) to obtain

$$\left\| R^{(K)} \right\|_{L^2(\mathbf{T})} \leq \varepsilon^{K-1} \left\| \theta_K \right\|_{L^2(\mathbf{T})}. \tag{3.109}$$

Using (3.109) we get

$$\left\| R^{(K)} \right\|_{L^2(\mathbf{T})} = \left\| R^{(K+2)} + \varepsilon^{K+2} u_{K+2}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) + \varepsilon^{K+1} u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})}$$

$$\leq \left\| R^{(K+2)} \right\|_{L^2(\mathbf{T})} + \varepsilon^{K+2} \left\| u_{K+2}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})} + \varepsilon^{K+1} \left\| u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})}$$

$$\leq \varepsilon^{K+1} \left(\varepsilon \left\| \theta_{K+2} \right\|_{L^2(\mathbf{T})} + \varepsilon \left\| u_{K+2}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})} + \left\| u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})} \right) \leq C^{(K)} \varepsilon^{K+1}. \tag{3.110}$$

The equality (3.107) also implies that

$$\varepsilon^2 \left\| \nabla R^{(K)} \right\|_{L^2(\mathbf{T})}^2 \leq \varepsilon^{K-1} \left\| \theta_K \right\|_{L^2(\mathbf{T})} \left\| R^{(K)} \right\|_{L^2(\mathbf{T})}.$$

Hence, taking into account (3.110), the following estimate holds

$$\left\| \nabla R^{(K)} \right\|_{L^2(\mathbf{T})} \leq \varepsilon^{K-1} \sqrt{\left\| \theta_K \right\|_{L^2(\mathbf{T})} C^{(K)}}.$$

Finally,

$$\left\| \nabla R^{(K)} \right\|_{L^2(\mathbf{T})} = \left\| \nabla R^{(K+1)} + \varepsilon^{K+1} \nabla u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \right\|_{L^2(\mathbf{T})}$$

$$\leq \left\|\nabla R^{(K+2)}\right\|_{L^2(\mathbf{T})} + \varepsilon^{K+1}\left\|\nabla u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\right\|_{L^2(\mathbf{T})}$$

$$\leq \varepsilon^K\left(\sqrt{\left\|\theta_{K+1}\right\|_{L^2(\mathbf{T})} C^{(K+1)}} + \varepsilon\left\|\nabla u_{K+1}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\right\|_{L^2(\mathbf{T})}\right) \leq \tilde{C}^{(K)}\varepsilon^K.$$

Thus, the estimates (3.105) and (3.106) have been proved. In particular, taking $K = 0$ in (3.105) results in

$$\|u^\varepsilon(\boldsymbol{x}) - u_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)\|_{L^2(\mathbf{T})} \leq C^{(0)}\varepsilon,$$

where $u_0(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) = u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x}, \tilde{\boldsymbol{y}})$. This recovers the two-scale convergence, but also gives a bound for the rate of convergence.

3.3.3 A non-local constitutive relation associated with the homogenised system

As we have established above, the sequence of solutions u^ε to the problems (3.2) two-scale converges to the sum $u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x}, \tilde{\boldsymbol{y}})$. This implies that u^ε converges to $u^{(1)} + \langle w\rangle$ weakly in $\left[L^2(\mathbf{T})\right]^3$ and therefore in the context of linear conductivity, the sequence of electric fields $\boldsymbol{e}^\varepsilon := \nabla u^\varepsilon$ converges in $\left[H^{-1}_{per}(\mathbf{T})\right]^3$, which is defined as the dual to the space $\left[H^1_{per}(\mathbf{T})\right]^3$, to the smooth vector function $\boldsymbol{e} = \nabla u^{(1)} + \nabla\langle w\rangle$. From the asymptotics for u^ε obtained in Section 3.3.2 we derive the following expressions for the electric field $\boldsymbol{e}^\varepsilon = (e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon)^\top$:

$$e_1^\varepsilon(\boldsymbol{x}) = \left(\varepsilon^{-1}w_{,y_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}}) + u^{(1)}_{,1}(\boldsymbol{x}) + w_{,x_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}}) + (u_1)_{,y_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon), \tag{3.111}$$

$$e_2^\varepsilon(\boldsymbol{x}) = \left(\varepsilon^{-1}w_{,y_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}}) + u^{(1)}_{,2}(\boldsymbol{x}) + w_{,x_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}}) + (u_1)_{,y_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon), \tag{3.112}$$

$$e_3^\varepsilon(\boldsymbol{x}) = \left(u^{(1)}_{,3}(\boldsymbol{x}) + w_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon). \tag{3.113}$$

Using formulas (3.111)–(3.113) and the definition (3.1) of the matrix $\left(A_{ij}(\boldsymbol{y})\right)$ we obtain expressions for the components of the current $j_k^\varepsilon := A_{kl}(\boldsymbol{x}/\varepsilon)e_l^\varepsilon$, $k = 1, 2, 3$ as follows

$$j_1^\varepsilon(\boldsymbol{x}) = \left(u^{(1)}_{,1}(\boldsymbol{x}) + (u_1)_{,y_1}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}})\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon), \tag{3.114}$$

$$j_2^\varepsilon(\boldsymbol{x}) = \left(u^{(1)}_{,2}(\boldsymbol{x}) + (u_1)_{,y_2}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}})\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon), \tag{3.115}$$

$$j_3^\varepsilon(\boldsymbol{x}) = \left(u^{(1)}_{,3}(\boldsymbol{x}) + w_{,x_3}(\boldsymbol{x}, \tilde{\boldsymbol{y}})\right)\Big|_{\tilde{\boldsymbol{y}}=\frac{\tilde{\boldsymbol{x}}}{\varepsilon}} + O(\varepsilon). \tag{3.116}$$

It is clear from the equalities (3.114)–(3.116) that the sequence of current field tensors $\boldsymbol{j}^\varepsilon = (j_1^\varepsilon, j_2^\varepsilon, j_3^\varepsilon)^\top$ strongly two-scale converges (see *e.g.* Zhikov [60]) to the vector

$$
\boldsymbol{j}_0(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} \left(u^{(1)}_{,1}(\boldsymbol{x}) + (u_1)_{,y_1}(\boldsymbol{x},\tilde{\boldsymbol{y}})\right)\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}}) \\ \left(u^{(1)}_{,2}(\boldsymbol{x}) + (u_1)_{,y_2}(\boldsymbol{x},\tilde{\boldsymbol{y}})\right)\chi_{\tilde{F}_0}(\tilde{\boldsymbol{y}}) \\ u^{(1)}_{,3}(\boldsymbol{x}) + w_{,x_3}(\boldsymbol{x},\tilde{\boldsymbol{y}}) \end{pmatrix}.
$$

Therefore, this sequence also converges weakly in $\left[L^2(\mathbf{T})\right]^3$ to the vector

$$
\boldsymbol{j} = \left\langle \boldsymbol{j}_0(\boldsymbol{x},\boldsymbol{y}) \right\rangle_y = \begin{pmatrix} \tilde{h}_{11}u^{(1)}_{,1} + \tilde{h}_{12}u^{(1)}_{,2} \\ \tilde{h}_{21}u^{(1)}_{,1} + \tilde{h}_{22}u^{(1)}_{,2} \\ u^{(1)}_{,3} + \langle w \rangle_{,3} \end{pmatrix}.
$$

Here we used formulas (3.88) and (3.59).

In this subsection we present the non-local homogenised constitutive relation, *i.e.* the relation between $\boldsymbol{j}$ and $\boldsymbol{e}$. To derive it, we employ the Fourier transform once again. The resulting relation is as follows

$$
\boldsymbol{j} = A^{hom}\boldsymbol{e} + \tilde{h}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left[\langle G \rangle_{\tilde{y},\tilde{y}'} \overset{x_3}{*} \Phi * \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} & 0 \\ \tilde{h}_{21} & \tilde{h}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{e} \right], \tag{3.117}
$$

Here, the $\mathbf{T}$-periodic kernel $\Phi(\boldsymbol{x})$ is the solution of the following equation

$$
-\tilde{h}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\langle G \rangle_{\tilde{y},\tilde{y}'} \overset{x_3}{*} \Phi \right) + \frac{\partial^2}{\partial x_3^2}\left(\langle G \rangle_{\tilde{y},\tilde{y}'} \overset{x_3}{*} \Phi \right) + \Phi = \delta, \qquad \boldsymbol{x} \in \mathbf{T}, \tag{3.118}
$$

where $\delta = \delta(\boldsymbol{x})$ is the Dirac's delta-function. In other words, the function $\Phi(\boldsymbol{x})$ is the fundamental solution of the integro-differential operator in the left-hand side of (3.118).

3.3.4 An example of the derived homogenised equations in the case of fibres with circular cross-section

In this subsection we derive explicit formulas (3.130) and (3.135) for the convolution kernel of the non-local homogenised operator in a particular case when the fibres have circular cross-section. To this end, we consider the following problem (*cf.* (3.60)–(3.61)):

$$
-\frac{\partial^2 G}{\partial y_1^2} - \frac{\partial^2 G}{\partial y_2^2} - \frac{\partial^2 G}{\partial x_3^2} = \delta(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}')\delta(x_3), \qquad \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}' \in D(0,a), \quad x_3 \in \mathbf{R}
$$

together with the boundary conditions

$$
G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3)|_{\tilde{y} \in C(0,a)} = 0,
$$

$$
G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3) \text{ is } x_3\text{-periodic with period } T
$$

where $D(0,a)$ and $C(0,a)$ are the disc and circumference of radius $a \in (0, \frac{1}{2})$ with the centre at the origin. Without loss of generality we can consider the case $T = 2\pi$.

We write the Green's function G as a Fourier series in x_3 as follows

$$G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m) \exp(imx_3), \tag{3.119}$$

where the Fourier coefficients $\hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)$ satisfy the following boundary value problems

$$-\frac{\partial^2 \hat{G}}{\partial y_1^2} - \frac{\partial^2 \hat{G}}{\partial y_2^2} + m^2 \hat{G} = \frac{1}{\sqrt{2\pi}} \delta(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}'), \qquad \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}' \in D(0,a), \tag{3.120}$$

$$\hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)|_{\tilde{\boldsymbol{y}} \in C(0,a)} = 0. \tag{3.121}$$

Rewriting these problems in polar coordinates $(r, \varphi), (r', \varphi')$ on $D(0,a)$ we arrive at the following equations

$$-\frac{\partial^2 \hat{G}}{\partial r^2} - \frac{1}{r}\frac{\partial \hat{G}}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \hat{G}}{\partial \varphi^2} + m^2 \hat{G} = \frac{1}{r\sqrt{2\pi}} \delta(r - r')\delta(\varphi - \varphi'), \quad r, r' \in (0,a), \ \varphi, \varphi' \in (0, 2\pi),$$

where the functions $\hat{G}(r, r', \varphi, \varphi', m)$ are sought to be 2π-periodic in φ, to satisfy the boundary condition $G(a, r, \varphi, \varphi') = 0$ and to be smooth at the singular point of the equation $r = 0$. Note that if we write the function $\hat{G}(r, r', \varphi, \varphi')$ as a Fourier series in φ:

$$\hat{G}(r, r', \varphi, \varphi', m) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{\hat{G}}(r, r', k, \varphi', m) \exp(ik\varphi),$$

then the average $\langle \hat{G} \rangle_{\tilde{y}, \tilde{y}'}$ of the function $\hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)$ with respect to $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{y}}'$ can be expressed in the following way

$$\left\langle \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m) \right\rangle_{\tilde{y}, \tilde{y}'} = \frac{1}{\pi^2 a^4} \int_0^{2\pi}\int_0^{2\pi}\int_0^{a}\int_0^{a} \hat{G}(r, r', \varphi, \varphi', m) \mathrm{d}r' \mathrm{d}r \mathrm{d}\varphi' \mathrm{d}\varphi =$$

$$= \frac{1}{\pi^2 \sqrt{2\pi} a^4} \sum_{k=-\infty}^{\infty} \int_0^{2\pi}\int_0^{2\pi}\int_0^{a}\int_0^{a} \hat{\hat{G}}(r, r', k, \varphi', m) \exp(ik\varphi) \mathrm{d}r' \mathrm{d}r \mathrm{d}\varphi' \mathrm{d}\varphi =$$

$$= \frac{\sqrt{2\pi}}{\pi^2 a^4} \int_0^{2\pi}\int_0^{a}\int_0^{a} \hat{\hat{G}}(r, r', 0, \varphi', m) \mathrm{d}r' \mathrm{d}r \mathrm{d}\varphi', \tag{3.122}$$

where for each $m \in \mathbf{Z}$ the function $\hat{\hat{G}}(r, r', 0, \varphi', m)$ satisfies the following equation

$$-\frac{d^2 \hat{\hat{G}}}{dr^2} - \frac{1}{r}\frac{d\hat{\hat{G}}}{dr} + m^2 \hat{\hat{G}} = \frac{1}{2\pi r} \delta(r - r'), \qquad r, r' \in (0,a), \tag{3.123}$$

with the boundary condition $\hat{\hat{G}}(a, r', 0, \varphi', m) = 0$ and so that $\hat{\hat{G}}$ is smooth at the point $r = 0$.

Let us first consider the case when $m \neq 0$. The problem (3.123) is an ordinary differential equation of second order and modified Bessel functions $I_0(mr)$ and $K_0(mr)$ form its fundamental system. We use the method of variation of arbitrary constant to find the solution of (3.123) that satisfies the imposed boundary condition at the point $r = a$ and smoothness condition at the point $r = 0$. As a result of this standard procedure we obtain the following formula for the function $\hat{\hat{G}}(r, r', 0, \varphi, m)$

$$\hat{\hat{G}}(r, r', 0, \varphi, m) = \frac{1}{2\pi}\Big(I_k(ma)\Big)^{-1} I_0(mr_<)\Big(I_0(ma)K_0(mr_>) - I_0(mr_>)K(ma)\Big), \tag{3.124}$$

where $r_< := \min\{r, r'\}$ and $r_> := \max\{r, r'\}$.

Using the formulas (3.122) and (3.124) we get

$$\Big\langle \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)\Big\rangle_{\tilde{y},\tilde{y}'} = \frac{2\sqrt{2\pi}}{\pi^2 a^4} \int\limits_0^a \int\limits_0^r \frac{I_0(mr')}{I_0(ma)}\Big(I_0(ma)K_0(mr) - I_0(mr)K(ma)\Big) r r' \mathrm{d}r' \mathrm{d}r$$

$$= \frac{2\sqrt{2\pi}}{\pi^2 a^4} \int\limits_0^a \frac{I_0(mr')}{I_0(ma)}\Big(I_0(ma)K_0(mr) - I_0(mr)K(ma)\Big) r \int\limits_0^r I_0(mr') r' \mathrm{d}r' \mathrm{d}r. \tag{3.125}$$

We use a well-known formula (see *e.g.* Gradshteyn & Ryzhik [25])

$$\int\limits_0^1 I_\nu(\alpha x) x^{\nu+1} dx = \alpha^{-1} I_{\nu+1}(\alpha), \qquad \alpha \in \mathbf{C},\ \operatorname{Re}\nu > -1 \tag{3.126}$$

with $\nu = 0$, to integrate with respect to r' in (3.125) and get

$$\Big\langle \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)\Big\rangle_{\tilde{y},\tilde{y}'} = \frac{2\sqrt{2\pi}}{\pi^2 a^4 m} \int\limits_0^a \frac{I_1(mr)}{I_0(ma)}\Big(I_0(ma)K_0(mr) - I_0(mr)K(ma)\Big) r^2 \mathrm{d}r$$

$$= \frac{2\sqrt{2\pi}}{\pi^2 am}\Big(I_0(ma)\Big)^{-1} \int\limits_0^1 I_1(max)\Big(I_0(ma)K_0(max) - I_0(max)K(ma)\Big) x^2 \mathrm{d}x.$$

Further, we apply formulas (3.143) and (3.145) (see Appendix D) with $\alpha = ma$ to obtain

$$\Big\langle \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', m)\Big\rangle_{\tilde{y},\tilde{y}'} = \frac{2\sqrt{2\pi}}{\pi^2 am}\Bigg(\frac{1}{2ma}\Big(\frac{1}{2} - K_1(ma)I_1(ma)\Big)$$

$$-\Big(I_0(ma)\Big)^{-1} K_0(ma)\frac{1}{2ma}\Big(I_1(ma)\Big)^2\Bigg) = \frac{\sqrt{2\pi}}{\pi^2 a^2 m^2}\Bigg(\frac{1}{2} - \frac{I_1(ma)}{I_0(ma)}\Big(K_0(ma)I_1(ma)$$

$$+K_1(ma)I_0(ma)\Big)\Bigg) = \frac{\sqrt{2\pi}}{\pi^2 a^2 m^2}\Big(\frac{1}{2} - \frac{I_1(ma)}{maI_0(ma)}\Big) = \frac{\sqrt{2\pi}}{2\pi^2 a^2 m^2}\frac{I_2(ma)}{I_0(ma)}. \tag{3.127}$$

Here, to get the first equality in (3.127) we use (3.148) with $\nu = 0$, and the second equality is due to the recurrent relation between Bessel functions with different indices as follows (see *e.g.* Gradshteyn & Ryzhik [25])

$$zI_{\nu-1}(z) - zI_{\nu+1}(z) = 2\nu I_\nu(z), \qquad \nu, z \in \mathbf{C},$$

which we apply with $\nu = 1$, $z = ma$.

In the case $m = 0$ the equation (3.123) takes the following form

$$-\frac{d^2\hat{\hat{G}}}{dr^2} - \frac{1}{r}\frac{d\hat{\hat{G}}}{dr} = \frac{1}{2\pi r}\delta(r - r'), \qquad r, r' \in (0, a), \tag{3.128}$$

which must be satisfied with the boundary condition $\hat{\hat{G}}(a, r', 0, \varphi', 0) = 0$ and smoothness condition at the point $r = 0$. Note that the functions 1 and $\log r$ form the fundamental system of (3.128). Using the method of variation of arbitrary constant once again it is not difficult to find the required solution

$$\hat{\hat{G}}(a, r', 0, \varphi', 0) = \frac{1}{2\pi}\log\left(\frac{a}{r_>}\right).$$

Integrating the last expression in accordance with the formula (3.122) we get

$$\left\langle \hat{G}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', 0) \right\rangle_{\tilde{y}, \tilde{y}'} = \frac{2\sqrt{2\pi}}{\pi^2 a^4}\int\limits_0^a\int\limits_0^r \log\left(\frac{a}{r}\right) r' r\, dr'\, dr = \frac{1}{4(2\pi)^{\frac{3}{2}}}. \tag{3.129}$$

Finally, using formulas (3.119), (3.127) and (3.129) we obtain the formula for the average of the Green's function as follows

$$\left\langle G(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}', x_3) \right\rangle_{\tilde{y}, \tilde{y}'} = \frac{1}{16\pi^2} + \frac{1}{2\pi^2 a^2}\sum_{m \in \mathbf{Z}\setminus\{0\}} \frac{I_2(ma)}{m^2 I_0(ma)}\exp(imx_3). \tag{3.130}$$

Let $\{\psi_j\}_{j=1}^{\infty}$ be an orthonormal basis in $L^2\Big(D(0, a)\Big)$. Clearly,

$$\delta(\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}') = \sum_{j=1}^{\infty} \psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')}.$$

Using the last identity we can rewrite (3.120) as

$$-\frac{\partial^2\hat{G}}{\partial y_1^2} - \frac{\partial^2\hat{G}}{\partial y_2^2} + m^2\hat{G} = \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{\infty}\psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')}, \qquad \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}' \in D(0, a). \tag{3.131}$$

We seek the solution to the problem (3.120)—(3.121) in the following form

$$\hat{G}(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}',m)=\sum_{j=1}^{\infty}c_j(m)\psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')},\tag{3.132}$$

where $\{\psi_j\}_{j=1}^{\infty}$ is the complete orthonormal set of eigenfunctions of the homogeneous Dirichlet problem for Laplacian in the disc $D(0,a)$. We denote λ_j the eigenvalue associated with ψ_j. Substituting (3.132) in the equation (3.131) and equating the coefficients in front of the terms with the same indices j, we arrive at an expression for the coefficients $c_j(m)$ as follows

$$c_j(m)=\frac{1}{\sqrt{2\pi}(\lambda_j+m^2)}.$$

Therefore, (3.132) takes the form

$$\hat{G}(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}',m)=\sum_{j=1}^{\infty}\frac{\psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')}}{\lambda_j+m^2}.$$

Hence, (3.119) implies

$$G(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}',x_3)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{\psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')}}{\lambda_j+m^2}\exp(imx_3).\tag{3.133}$$

Changing the order of summation in (3.133) we get

$$G(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}',x_3)=\frac{1}{2\pi}\sum_{j=1}^{\infty}\psi_j(\tilde{\boldsymbol{y}})\overline{\psi_j(\tilde{\boldsymbol{y}}')}\sum_{m=-\infty}^{\infty}\frac{\exp(imx_3)}{\lambda_j+m^2}.$$

Averaging the last formula with respect to $\tilde{\boldsymbol{y}}$ and $\tilde{\boldsymbol{y}}'$ we obtain the following equality

$$\Big\langle G(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}',x_3)\Big\rangle_{\tilde{\boldsymbol{y}},\tilde{\boldsymbol{y}}'}=\frac{1}{2\pi}\sum_{j=1}^{\infty}\Big|\Big\langle\psi_j(\tilde{\boldsymbol{y}})\Big\rangle_{\tilde{\boldsymbol{y}}}\Big|^2\sum_{m=-\infty}^{\infty}\frac{\exp(imx_3)}{\lambda_j+m^2}.\tag{3.134}$$

The system $\{\psi_j\}_{j=1}^{\infty}$ in our particular case is well known. Upon passing to polar coordinates and re-indexing the elements of the system appropriately, it can be written in the following way

$$\psi_{kl}^{\pm}(r,\varphi)=\frac{J_k(\sqrt{\lambda_{kl}}r)\exp(\pm ik\varphi)}{a\sqrt{\pi}J_{k+1}(\sqrt{\lambda_{kl}}a)},\quad k,l\in\mathbf{N};\quad \psi_{0l}(r,\varphi)=\frac{J_0(\sqrt{\lambda_{0l}}r)}{a\sqrt{\pi}J_1(\sqrt{\lambda_{0l}}a)},\quad l\in\mathbf{N}.$$

In the last formula, J_k are standard Bessel functions of the first kind and $\lambda_{kl}=a^{-2}(z_{k,l})^2$, where $z_{k,l}$ is the l-th zero of J_k, are eigenvalues associated with $\psi_{kl}^{\pm}$.

Also, the following formula, which can be found in *e.g.* Gradshteyn & Ryzhik [25], is not difficult to verify

$$\sum_{m=-\infty}^{\infty} \frac{\exp(imx_3)}{\lambda_j + m^2} = \frac{\pi}{\sqrt{\lambda_j}} \frac{\cosh\Big(\sqrt{\lambda_j}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_j})}$$

Note that the functions $\psi^{\pm}_{kl}$, $k \in \mathbf{N}$ have zero mean over the disc $D(0,a)$, and therefore the formula (3.134) can be specified as follows

$$\Big\langle G(\tilde{y}, \tilde{y}', x_3)\Big\rangle_{\tilde{y},\tilde{y}'} = \frac{1}{2\pi}\sum_{l=1}^{\infty}\left|\Big\langle \psi_{0l}\Big\rangle_{D(0,a)}\right|^2 \frac{\pi}{\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}$$

$$= \frac{1}{2\pi}\sum_{l=1}^{\infty}\left(\frac{1}{\pi a^2}\frac{1}{a\sqrt{\pi}J_1(\sqrt{\lambda_{0l}}a)}\int_0^{2\pi}\int_0^a J_0(\sqrt{\lambda_{0l}}r) r\,dr\,d\varphi\right)^2 \frac{\pi}{\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}$$

$$= \frac{2}{\pi a^2}\sum_{l=1}^{\infty}\left(\frac{1}{J_1(\sqrt{\lambda_{0l}}a)}\int_0^1 J_0(\sqrt{\lambda_{0l}}ax) x\,dx\right)^2 \frac{1}{\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}$$

$$= \frac{2}{\pi a^2}\sum_{l=1}^{\infty}\left(\frac{1}{J_1(\sqrt{\lambda_{0l}}a)}\frac{J_1(\sqrt{\lambda_{0l}}a)}{\sqrt{\lambda_{0l}}a}\right)^2 \frac{1}{\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}$$

$$= \frac{2}{\pi a^4}\sum_{l=1}^{\infty}\frac{1}{\lambda_{0l}\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}. \tag{3.135}$$

Note that to get the penultimate equality in (3.135), we use the following formula (see *e.g.* Gradshteyn & Ryzhik [25]; *cf.* (3.126))

$$\int_0^1 J_\nu(\alpha x) x^{\nu+1} dx = \alpha^{-1} J_{\nu+1}(\alpha), \qquad \alpha \in \mathbf{C},\ \operatorname{Re}\nu > -1$$

setting $\nu = 0$.

The formulas (3.130) and (3.135) deliver explicit representations for the kernel of the non-local homogenised operator. They give two different expressions for the same function $\Big\langle G(\tilde{y}, \tilde{y}', x_3)\Big\rangle_{\tilde{y},\tilde{y}'}$. Therefore, we can equate these expressions to get the following identity with respect to x_3

$$\frac{1}{16\pi^2} + \frac{1}{2\pi^2 a^2}\sum_{m\in\mathbf{Z}\setminus\{0\}} \frac{I_2(ma)}{m^2 I_0(ma)}\exp(imx_3) = \frac{2}{\pi a^4}\sum_{l=1}^{\infty}\frac{1}{\lambda_{0l}\sqrt{\lambda_{0l}}} \frac{\cosh\Big(\sqrt{\lambda_{0l}}(\pi - |x_3|)\Big)}{\sinh(\pi\sqrt{\lambda_{0l}})}.$$

We integrate this identity over the interval $(-\pi, \pi)$ to obtain the following curious formula

$$\frac{4}{\pi a^4}\sum_{l=1}^{\infty}\frac{1}{\lambda_{0l}^2} = \frac{1}{8\pi},$$

It can also be written as

$$\sum_{l=1}^{\infty} \frac{1}{z_{0,l}^4} = \frac{1}{32},$$

which is new to the best of our knowledge. (We were unable to find a relation of this type for zeros $z_{0,l}$ of the Bessel function J_0 in Gradshteyn & Ryzhik [25] or other reference books on special functions.)

Discussion

In this chapter we propose a study of a linear periodic rapidly oscillating problem set on a mixture of several materials whose homogenised limit exhibits a spatially non-local effect. The problem derives from an equation of "double porosity" type, which was previously mathematically investigated by Zhikov [60] and others. The key feature of the double porosity models is that the small parameter $\varkappa$ of contrast between the constituent phases and the small period ε of the coefficients of the original heterogeneous equation are related by the formula $\varkappa \sim \varepsilon^2$.

Using this model situation as a basis, we give an exposition of several possible ways to treat problems with non-local effects. The well-known two-scale convergence method for passing to the limit in rapidly oscillating problems can be used to find the non-local homogenised limit and adequately describe the convergence of solutions. Furthermore, the results obtained using the above method can be further substantiated and amended by applying another classical method for studying problems with small parameters, the method of asymptotic expansion. In particular, the high-contrast version of the Poincaré inequality proved in this chapter enables us to get the natural remainder estimates for the double-series two-scale asymptotic expansion in the case of periodic mixtures of several materials with highly contrasting properties.

Using a combination of these two methods we have studied the model situation of a mixture of two conducting materials one of which is included in the other in the shape of fibres having conductivity of order one in the direction along the fibres and very small conductivities (of order ε^2) in the directions orthogonal to the fibres. The resulting limiting current and electric fields are proved to be linked by a non-local constitutive law involving non-locality in the direction with conductivity of order one. The kernel of the corresponding convolution operator is given explicitly via the Green's function of the fibres and therefore our results can be used for numerical implementation.

Appendix A: Two-scale convergence: definition and basic properties

Here we review some of the facts about the two-scale convergence, due to Nguetseng [35], Allaire [4] and Zhikov [60].

1. Definition. A sequence of functions $u^\varepsilon(\boldsymbol{x}) \in L^2(\mathbf{T})$ is said to two-scale converge to the function $u(\boldsymbol{x},\boldsymbol{y}) \in L^2(\mathbf{T}\times Q)$ if for any test function $\psi(\boldsymbol{x},\boldsymbol{y}) \in C(\mathbf{T}\times Q)$ the following convergence holds

$$\int_{\mathbf{T}} u^\varepsilon(\boldsymbol{x})\psi(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\mathrm{d}\boldsymbol{x} \overset{\varepsilon\to\infty}{\longrightarrow} \int_{\mathbf{T}}\int_{Q} u(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}.$$

If $u^\varepsilon(\boldsymbol{x})$ two-scale converges to $u(\boldsymbol{x},\boldsymbol{y})$, we write $u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} u(\boldsymbol{x},\boldsymbol{y})$.

2. Compactness property. If a sequence of functions $u^\varepsilon(\boldsymbol{x}) \in L^2(\mathbf{T})$ is bounded in $L^2(\mathbf{T})$ then there exist a subsequence $u^{\varepsilon_j}(\boldsymbol{x})$ and a function $u(\boldsymbol{x},\boldsymbol{y}) \in L^2(\mathbf{T}\times Q)$ (possibly depending on the subsequence chosen), such that $u^{\varepsilon_j}(\boldsymbol{x}) \overset{2}{\rightharpoonup} u(\boldsymbol{x},\boldsymbol{y})$.

3. If the sequences $u^\varepsilon(\boldsymbol{x}) \in H^1(\mathbf{T})$ and $\varepsilon\nabla u^\varepsilon(\boldsymbol{x})$ are bounded in $L^2(\mathbf{T})$ and $\left[L^2(\mathbf{T})\right]^3$ respectively, then there exists a function $u(\boldsymbol{x},\boldsymbol{y}) \in L^2\left(\mathbf{T}, H^1_{per}(Q)\right)$ such that, up to a subsequence, $u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} u(\boldsymbol{x},\boldsymbol{y})$ and $\varepsilon\nabla u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \nabla_y u(\boldsymbol{x},\boldsymbol{y})$.

4. If $H^1(\mathbf{T}) \ni u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} u(\boldsymbol{x}) \in H^1(\mathbf{T})$ and $\nabla u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) \in \left[L^2(\mathbf{T}\times Q)\right]^3$ then the difference $\boldsymbol{v}(\boldsymbol{x},\boldsymbol{y}) - \nabla u(\boldsymbol{x})$ is a potential vector function of $\boldsymbol{y}$ for any $\boldsymbol{x} \in \mathbf{T}$.[6]

5. Multiplication by a bounded function. If $u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} u(\boldsymbol{x},\boldsymbol{y})$ and $\varphi(\boldsymbol{y}) \in L^\infty(Q)$ then $u^\varepsilon(\boldsymbol{x})\varphi(\boldsymbol{x}/\varepsilon) \overset{2}{\rightharpoonup} u(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})$.

Appendix B: The restriction of the limiting function to the hard phase $u^{(1)}(\boldsymbol{x})$ belongs to the space $H^1_{per}(\mathbf{T})$

In this section we are going to prove that the restriction of the limiting function $u(\boldsymbol{x},\boldsymbol{y})$ from (3.11) to the hard phase belongs to the Sobolev space $H^1_{per}(\mathbf{T})$.

Denote $\mathrm{d}\mu_1^\varepsilon := \mathrm{d}\boldsymbol{x}|_{F_1^\varepsilon}$. Then the relation $\chi_1(\varepsilon^{-1}\boldsymbol{x})u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} \chi_1(\boldsymbol{y})u^{(1)}(\boldsymbol{x})$ can be rewritten as

$$L^2(\mathbf{T},\mathrm{d}\mu_1^\varepsilon) \ni u^\varepsilon(\boldsymbol{x}) \overset{2}{\rightharpoonup} u^{(1)}(\boldsymbol{x}) \in L^2(\mathbf{T},\mathrm{d}\mu_1). \tag{3.136}$$

Notice, that for any $\varepsilon > 0$ the function u^ε is the limit of a sequence of smooth $\mathbf{T}$-periodic functions U_n^ε as $n \to \infty$ in the $H^1(\mathbf{T})$-norm. Furthermore, for any function $\phi \in C^\infty_{per}(\mathbf{T})$ and vector function $\boldsymbol{h} \in V_{sol}(Q,\mathrm{d}\mu_1)$[7] and for any $\varepsilon > 0$, $n \in \mathbf{N}$ the

[6]By definition, the set of potential vectors is the closure of the set $\left\{\nabla u : u \in C^\infty_{per}(Q)\right\}$ in the Lebesgue space $\left[L^2(Q)\right]^3$.

[7]The set $V_{sol}(Q,\mathrm{d}\mu_1)$ of solenoidal vectors is defined as the orthogonal complement of the set $\left\{\nabla u : u \in C^\infty_{per}(Q)\right\}$ in the space $\left[L^2(Q,\mathrm{d}\mu_1)\right]^3$.

following formula holds

$$\int_{\mathbf{T}} \phi(\boldsymbol{x})\nabla U_n^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon = \int_{\mathbf{T}} \nabla\Big(U_n^\varepsilon(\boldsymbol{x})\phi(\boldsymbol{x})\Big) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon$$

$$= \int_{\mathbf{T}} U_n^\varepsilon(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon = -\int_{\mathbf{T}} U_n^\varepsilon(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon.$$

Here we used integration by parts, periodicity of the functions involved and the fact that $\boldsymbol{h}(\boldsymbol{y})$ is a solenoidal vector function. Passing to the limit in the first and last parts of this formula as $n \to \infty$ we conclude that

$$\int_{\mathbf{T}} \phi(\boldsymbol{x})\nabla u^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon = -\int_{\mathbf{T}} u^\varepsilon(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{h}(\varepsilon^{-1}\boldsymbol{x})\mathrm{d}\mu_1^\varepsilon. \tag{3.137}$$

It follows from the inequality (3.10) that ∇u^ε is bounded in $\left[L^2(\mathbf{T}, \mathrm{d}\mu_1^\varepsilon)\right]^3$ and therefore, there exists a vector function $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \in \left[L^2(\mathbf{T} \times Q, \mathrm{d}\boldsymbol{x} \times \mathrm{d}\mu_1^\varepsilon)\right]^3$ such that up to a subsequence

$$\nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}).$$

Using this and (3.136) we pass to the limit in (3.137) and arrive at the following identity

$$\int_{\mathbf{T}}\int_{Q} \phi(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{h}(\boldsymbol{y})\mathrm{d}\mu_1\mathrm{d}\boldsymbol{x} = -\int_{\mathbf{T}}\int_{Q} u^{(1)}(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) \cdot \boldsymbol{h}(\boldsymbol{y})\mathrm{d}\mu_1\mathrm{d}\boldsymbol{x}.$$

We rewrite the right-hand side of the last formula to get

$$\int_{\mathbf{T}}\int_{Q} \phi(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{h}(\boldsymbol{y})\mathrm{d}\mu_1\mathrm{d}\boldsymbol{x} = -\int_{\mathbf{T}} u^{(1)}(\boldsymbol{x})\nabla\phi(\boldsymbol{x}) \cdot \langle\boldsymbol{h}\rangle\mathrm{d}\boldsymbol{x}.$$

Thus, denoting $\langle\boldsymbol{h}\rangle =: \boldsymbol{a} \in \mathbf{R}^3$ we conclude that $\boldsymbol{a} \cdot \nabla u^{(1)} \in L^2(\mathbf{T})$ and

$$\boldsymbol{a} \cdot \nabla u^{(1)}(\boldsymbol{x}) = \int_{Q} \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{h}(\boldsymbol{y})\mathrm{d}\mu_1.$$

From the fact that the measure $\mathrm{d}\mu_1$ is non-degenerate, we deduce that for any $\boldsymbol{a} \in \mathbf{R}^3$ there exists $\boldsymbol{h} \in V_{sol}(Q, \mathrm{d}\mu_1)$ such that $\langle\boldsymbol{h}\rangle = \boldsymbol{a}$, (for a proof of this, see Zhikov [60]) and therefore it is proved that for any $\boldsymbol{a} \in \mathbf{R}^3$ the expression $\boldsymbol{a} \cdot \nabla u^{(1)}$ belongs to the Lebesgue class $L^2(\mathbf{T})$. This results in $\nabla u^{(1)} \in \left[L^2(\mathbf{T})\right]^3$ and therefore $u^{(1)} \in H^1(\mathbf{T})$.

Moreover, $u^{(1)}$ is the limit with respect to $H^1(\mathbf{T})$-norm of a sequence of smooth periodic functions, due to the fact that the identities

$$\int_{\mathbf{T}} \phi(\boldsymbol{x})u^{(1)}_{,i}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_{\mathbf{T}} u^{(1)}(\boldsymbol{x})\phi_{,i}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \quad i = 1, 2, 3$$

hold for all $\phi \in C^{\infty}_{per}(\mathbf{T})$. This can be proved easily by considering the space $H^1(\mathcal{T})$, where $\mathcal{T}$ is a torus obtained in a natural way from the periodicity cell $\mathbf{T}$, and using partition of unity on $\mathcal{T}$.

Appendix C: The proof of the equality (3.17)

In our proof of the equality (3.17) we follow the argument of Zhikov [60].

Let us consider test functions of the form

$$\psi(\boldsymbol{x}) = \psi^{\varepsilon}(\boldsymbol{x}) = \varepsilon\phi(\boldsymbol{x})h(\varepsilon^{-1}\tilde{\boldsymbol{x}}),$$

where $\phi(\boldsymbol{x}) \in C^{\infty}_{per}(\mathbf{T})$ and $h \in C^{\infty}_{per}(Q_2)$. Then

$$\nabla\psi^{\varepsilon}(\boldsymbol{x}) = \phi(\boldsymbol{x})\nabla_y h(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}\tilde{x}} + \varepsilon h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\nabla\phi(\boldsymbol{x}).$$

We substitute $\psi^{\varepsilon}(\boldsymbol{x})$ into the original integral identity (3.3) and obtain the following equality

$$\int\limits_{\mathbf{T}\cap F_1^{\varepsilon}} \phi(\boldsymbol{x})\nabla u^{\varepsilon}(\boldsymbol{x}) \cdot \nabla_y h(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}\tilde{x}}\mathrm{d}\boldsymbol{x} + \int\limits_{\mathbf{T}\cap F_1^{\varepsilon}} h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\nabla u^{\varepsilon}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

$$+\varepsilon^2 \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} \Big[u^{\varepsilon}_{,1}(\boldsymbol{x})\Big(\phi(\boldsymbol{x})h_{,1}(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}\tilde{x}} + \varepsilon h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\phi_{,1}(\boldsymbol{x})\Big)$$

$$+u^{\varepsilon}_{,2}(\boldsymbol{x})\Big(\phi(\boldsymbol{x})h_{,2}(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}\tilde{x}} + \varepsilon h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\phi_{,1}(\boldsymbol{x})\Big)\Big]\mathrm{d}\boldsymbol{x} + \varepsilon \int\limits_{\mathbf{T}\cap F_0^{\varepsilon}} u^{\varepsilon}_{,3}(\boldsymbol{x})h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\phi_{,3}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

$$+\lambda\varepsilon \int\limits_{\mathbf{T}} u^{\varepsilon}(\boldsymbol{x})\phi(\boldsymbol{x})h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x} = \int\limits_{\mathbf{T}} f(\boldsymbol{x},\varepsilon^{-1}\boldsymbol{x})\phi(\boldsymbol{x})h(\varepsilon^{-1}\tilde{\boldsymbol{x}})\mathrm{d}\boldsymbol{x}.$$

Using *a priori* estimates found above, it is easy to see that in the last identity all terms but the first one vanish as $\varepsilon \to 0$. Hence the first term vanishes as well, *i.e.*,

$$\int\limits_{\mathbf{T}\cap F_1^{\varepsilon}} \phi(\boldsymbol{x})\nabla u^{\varepsilon}(\boldsymbol{x}) \cdot \nabla_y h(\tilde{\boldsymbol{y}})|_{\tilde{y}=\varepsilon^{-1}\tilde{x}}\mathrm{d}\boldsymbol{x} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.138}$$

Denote $\mathrm{d}\mu_1^{\varepsilon} := \mathrm{d}\boldsymbol{x}|_{F_1^{\varepsilon}}$. It is proved by Zhikov [60] that due to boundedness of the sequence ∇u^{ε} in the space $L^2(\mathbf{T}, \mathrm{d}\mu_1^{\varepsilon})$, the following convergence holds (*cf.* Appendix A)

$$\nabla u^{\varepsilon}(\boldsymbol{x}) \overset{2}{\rightharpoonup} \nabla u^{(1)}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) \quad \text{with respect to } \mathrm{d}\mu_1, \tag{3.139}$$

where $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) \in L^2\Big(\mathbf{T}, V_{pot}(Q, \mathrm{d}\mu_1)\Big)$[8]. In particular,

$$\nabla u^{\varepsilon}(\boldsymbol{x}) \rightharpoonup \int\limits_Q \Big(\nabla u^{(1)}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\Big)\mathrm{d}\mu_1 \;\text{ in } \Big[L^2(\mathbf{T})\Big]^3. \tag{3.140}$$

Due to (3.139),

$$\int\limits_{\mathbf{T}} \phi(\boldsymbol{x})\nabla u^{\varepsilon}(\boldsymbol{x}) \cdot \nabla_y h(\tilde{\boldsymbol{y}})|_{y=\varepsilon^{-1}x}\mathrm{d}\mu_1^{\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} \int\limits_{\mathbf{T}}\int\limits_Q \Big(\nabla u^{(1)}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\Big) \cdot \nabla_y h(\tilde{\boldsymbol{y}})\phi(\boldsymbol{x})\mathrm{d}\mu_1\mathrm{d}\boldsymbol{x},$$

and in view of (3.138),

$$\int\limits_{\mathbf{T}}\int\limits_Q \Big(\nabla u^{(1)}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\Big) \cdot \nabla_y h(\tilde{\boldsymbol{y}})\phi(\boldsymbol{x})\mathrm{d}\mu_1\mathrm{d}\boldsymbol{x} = 0$$

for arbitrary $\phi(\boldsymbol{x}) \in C^{\infty}_{per}(\mathbf{T})$ and $h \in C^{\infty}_{per}(Q_2)$. Thus,

$$\int\limits_Q \Big(\nabla u^{(1)}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\Big) \cdot \nabla_y h(\tilde{\boldsymbol{y}})\mathrm{d}\mu_1 = 0$$

for any $h \in C^{\infty}_{per}(Q_2)$, or rewriting it in a more suitable form,

$$\int\limits_{Q_2} \left(\nabla_{\tilde{x}} u^{(1)}(\boldsymbol{x}) + \Big\langle \tilde{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3}\right) \cdot \nabla_{\tilde{y}} h(\tilde{\boldsymbol{y}})\mathrm{d}\tilde{\mu}_1 = 0, \tag{3.141}$$

where $\tilde{\mu}_1 := \mathrm{d}\tilde{\boldsymbol{y}}|_{\tilde{F}_1 \cap Q_2}$. Taking into account (3.140) we get

$$\nabla u^{\varepsilon}(\boldsymbol{x}) \rightharpoonup \left(\int\limits_{Q_2} \left(\nabla_{\tilde{x}} u^{(1)}(\boldsymbol{x}) + \Big\langle \tilde{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3}\right)\mathrm{d}\tilde{\mu}_1, \mathfrak{f}_1 u^{(1)}_{,3}(\boldsymbol{x}) + \int\limits_{Q_2}\Big\langle r_3(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3}\mathrm{d}\tilde{\mu}_1\right).$$

But $\Big\langle r_3(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3} = 0$, due to the fact that $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) \in L^2\Big(\mathbf{T}, V_{pot}(Q, \mathrm{d}\mu_1)\Big)$ and therefore the spaces $V_{pot}(Q, \mathrm{d}\mu_1)$ and $V_{pot}(Q, \mathrm{d}\tilde{\mu}_1 \times \mathrm{d}y_3)$ coincide. Hence,

$$\nabla u^{\varepsilon}(\boldsymbol{x}) \rightharpoonup \left(\int\limits_{Q_2} \left(\nabla_{\tilde{x}} u^{(1)}(\boldsymbol{x}) + \Big\langle \tilde{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{y})\Big\rangle_{y_3}\right)\mathrm{d}\tilde{\mu}_1, \mathfrak{f}_1 u^{(1)}_{,3}(\boldsymbol{x})\right)$$

$$= \Big(A^{hom}_{2D}\nabla_{\tilde{x}} u^{(1)}(\boldsymbol{x}), \mathfrak{f}_1 u^{(1)}_{,3}(\boldsymbol{x})\Big), \tag{3.142}$$

where the matrix A^{hom}_{2D} is given by the formula (3.7). The last equality in (3.142) is due to (3.141).

[8] By definition, the set $V_{pot}(Q, \mathrm{d}\mu_1)$ of potential vectors is the closure of the set $\Big\{\nabla u : u \in C^{\infty}_{per}(Q)\Big\}$ in the space $\Big[L^2(Q, \mathrm{d}\mu_1)\Big]^3$.

Appendix D: Two useful identities for Bessel functions

In this appendix we prove two identities we derived for use in Section 3.3.4.

Lemma 1.

The following formula holds

$$\int_0^1 I_0(\alpha x)I_1(\alpha x)x^2dx = \frac{1}{2\alpha}\Big(I_1(\alpha)\Big)^2, \qquad \alpha \in \mathbf{C}. \tag{3.143}$$

Proof:

Using a well-known formula (see *e.g.* Gradshteyn & Ryzhik [25])

$$\left(\frac{d}{zdz}\right)^n \Big\{z^\nu I_\nu(z)\Big\} = z^{\nu-n}I_{\nu-n}(z), \qquad z,\nu \in \mathbf{C},\ n \in \mathbf{N}$$

for $z = \alpha x$, $n = 1$, $\nu = 1$ and integrating by parts we get

$$\int_0^1 I_0(\alpha x)I_1(\alpha x)x^2dx = \int_0^1 \frac{d}{\alpha x d(\alpha x)}\Big\{\alpha x I_1(\alpha x)\Big\}I_1(\alpha x)x^2dx$$

$$= \alpha^{-1}\int_0^1 \frac{d}{dx}\Big\{xI_1(\alpha x)\Big\}I_1(\alpha x)xdx$$

$$= \alpha^{-1}\Big[\Big(xI_1(\alpha x)\Big)^2\Big]_0^1 - \alpha^{-1}\int_0^1 \frac{d}{dx}\Big\{xI_1(\alpha x)\Big\}I_1(\alpha x)xdx$$

$$= \alpha^{-1}\Big(I_1(\alpha)\Big)^2 - \int_0^1 \frac{d}{\alpha x d(\alpha x)}\Big\{\alpha x I_1(\alpha x)\Big\}I_1(\alpha x)x^2dx$$

$$= \alpha^{-1}\Big(I_1(\alpha)\Big)^2 - \int_0^1 I_0(\alpha x)I_1(\alpha x)x^2dx. \tag{3.144}$$

Therefore the first and the last expressions in (3.144) are equal, which gives us the required result. □

Lemma 2.

The following formula holds

$$\int_0^1 K_0(\alpha x)I_1(\alpha x)x^2dx = \frac{1}{2\alpha}\left(\frac{1}{2} - K_1(\alpha)I_1(\alpha)\right), \qquad \alpha \in \mathbf{C}. \tag{3.145}$$

Proof:

The first part of the proof is very similar to the proof of the Lemma 1. Namely, we use the formula (see *e.g.* Gradshteyn & Ryzhik [25])

$$\left(\frac{d}{zdz}\right)^n \Big\{z^\nu K_\nu(z)\Big\} = (-1)^n z^{\nu-n} I_{\nu-n}(z), \qquad z,\nu \in \mathbf{C},\ n \in \mathbf{N}$$

for $z = \alpha x$, $n = 1$, $\nu = 1$, and integrate by parts. Thus, we obtain the following chain of equalities

$$\int_0^1 K_0(\alpha x) I_1(\alpha x) x^2 dx = -\int_0^1 \frac{d}{\alpha x d(\alpha x)}\Big\{\alpha x K_1(\alpha x)\Big\} I_1(\alpha x) x^2 dx$$

$$= -\alpha^{-1} \int_0^1 \frac{d}{dx}\Big\{x K_1(\alpha x)\Big\} I_1(\alpha x) x dx$$

$$= -\alpha^{-1}\Big[K_1(\alpha x) I_1(\alpha x) x^2\Big]_0^1 + \alpha^{-1} \int_0^1 K_1(\alpha x) x \frac{d}{dx}\Big\{x I_1(\alpha x)\Big\} dx$$

$$= -\alpha^{-1} K_1(\alpha) I_1(\alpha) + \int_0^1 I_1(\alpha x) x^2 \frac{d}{\alpha x d(\alpha x)}\Big\{\alpha x I_1(\alpha x)\Big\} dx$$

$$= -\alpha^{-1} K_1(\alpha) I_1(\alpha) + \int_0^1 K_1(\alpha x) I_0(\alpha x) x^2 dx. \tag{3.146}$$

The equality of the first and the last expressions in (3.146) implies

$$\int_0^1 \Big(K_0(\alpha x) I_1(\alpha x) - K_1(\alpha x) I_0(\alpha x)\Big) x^2 dx = -\frac{1}{\alpha} K_1(\alpha) I_1(\alpha). \tag{3.147}$$

On the other hand, the following formula holds (see *e.g.* Gradshteyn & Ryzhik [25])

$$K_\nu(\alpha x) I_{\nu+1}(\alpha x) + K_{\nu+1}(\alpha x) I_\nu(\alpha x) = \frac{1}{\alpha x}, \tag{3.148}$$

which for $\nu = 0$ implies by integration

$$\int_0^1 \Big(K_0(\alpha x) I_1(\alpha x) + K_1(\alpha x) I_0(\alpha x)\Big) x^2 dx = \frac{1}{2\alpha}. \tag{3.149}$$

Adding (3.147) and (3.149) together we finally obtain the required formula (3.145). □

Chapter 4

Interrelations between the higher-gradient and non-local effects

In this chapter we make some concluding remarks to summarize the findings of Chapter 3 and their relation to the results of Chapter 1. We will use the case of linearised elasticity as a model example, considering anti-plane shear as in Chapter 1. This chapter aims in a sense at providing some unifying interpretation of both "higher-gradient" and "non-local" effects studied in the previous chapters.

4.1 Non-locality is a generic property of periodic heterogeneous media

In this section we show that for fixed cell size ε the relations between the "ensemble mean" stresses and strains are non-local.

As before, we consider the equilibrium equation for a linear ε-periodic (ε is a small but fixed positive parameter) medium subjected to a $\mathbf{T}$-periodic body force f having zero mean over $\mathbf{T} = [-T, T]^2$, as follows (see Section 1.1.1)

$$-\left(A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)u_{,i}^{\varepsilon,\varkappa}\right)_{,j} = f(\boldsymbol{x}), \quad \int\limits_{\mathbf{T}} u^{\varepsilon,\varkappa}\mathrm{d}\boldsymbol{x} = 0. \tag{4.1}$$

Here, the Q-periodic ($Q = [0,1]^2$) constitutive matrix $\left(A_{ij}^{\varkappa}(\boldsymbol{y})\right)$ depends on a parameter $\varkappa$ (*e.g.* contrast), which is of no importance in this section. We denote the Green's function of the problem (4.1) in $\mathbf{T}$ by $G^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')$. It is the solution to the following problem

$$-\left(A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)G_{,x_i}^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\right)_{,x_j} = \delta(\boldsymbol{x}-\boldsymbol{x}') - \frac{1}{|\mathbf{T}|}, \quad \int\limits_{\mathbf{T}} G^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{x} = 0 \text{ for every } \boldsymbol{x}' \in \mathbf{T}.$$

Thus, the solution to the problem (4.1) for an arbitrary fixed positive ε is given by the following formula

$$u^{\varepsilon}(\boldsymbol{x}) = \int\limits_{\mathbf{T}} G^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')f(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'.$$

Accordingly, the strain tensor of the medium is as follows

$$e_j^{\varepsilon,\varkappa}(\boldsymbol{x}) = \frac{1}{2}u_{,j}^{\varepsilon,\varkappa}(\boldsymbol{x}) = \frac{1}{2}\int\limits_{\mathbf{T}} G_{,x_j}^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')f(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}' = -\frac{1}{2}\int\limits_{\mathbf{T}} G^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')f_{,j}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'. \quad (4.2)$$

As in Section 1.2.2, we consider a family (an "ensemble") of problems (4.1) with "$\boldsymbol{\zeta}$-translated" periodic media $A^{\varkappa}(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta})$, $\boldsymbol{\zeta}\in Q$. This represents a simple random medium with probability measure being the Lebesgue measure $\mathrm{d}\boldsymbol{\zeta}$ on the space of realisations Q. Performing the translation averaging (the "ensemble averaging") introduced in Section 1.2.2 (see also Section 1.3.1), we consider the mean strain

$$\bar{e}_j^{\varepsilon,\varkappa}(\boldsymbol{x}) = \int\limits_{Q} e_j^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x})\mathrm{d}\boldsymbol{\zeta},$$

for which using the formula (4.2) we obtain

$$\bar{e}_j^{\varepsilon,\varkappa}(\boldsymbol{x}) = -\frac{1}{2}\int\limits_{\mathbf{T}} \Big\langle G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle_{\zeta} f_{,j}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'. \quad (4.3)$$

($G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')$ is the Green's function for the ζ-translated medium.)

The stress in the medium under consideration is given by the following formula

$$\sigma_i^{\varepsilon,\varkappa}(\boldsymbol{x}) = A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)u_{,j}^{\varepsilon,\varkappa}(\boldsymbol{x}) = -A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)\int\limits_{\mathbf{T}} G^{\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')f_{,j}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}',$$

and therefore the mean stress is

$$\bar{\sigma}_i^{\varepsilon,\varkappa}(\boldsymbol{x}) = \int\limits_{Q} \sigma_j^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x})\mathrm{d}\boldsymbol{\zeta} = -\int\limits_{Q} A_{ij}^{\zeta,\varkappa}(\boldsymbol{x}/\varepsilon)\int\limits_{\mathbf{T}} G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')f_{,j}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{\zeta}$$

$$= -\int\limits_{\mathbf{T}} \Big\langle A_{ij}^{\zeta,\varkappa}(\boldsymbol{x}/\varepsilon)G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle f_{,j}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'. \quad (4.4)$$

We next show that the right-hand sides of (4.3) and (4.4) are convolution operators. To this end notice that the "translated" Green's function $G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')$ solves the following problem

$$-\Big(A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta})u_{,x_i}(\boldsymbol{x},\boldsymbol{x}')\Big)_{,x_j} = \delta(\boldsymbol{x}-\boldsymbol{x}') - \frac{1}{|\mathbf{T}|}, \quad \int\limits_{\mathbf{T}} u(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{x} = 0 \ \text{ for every } \boldsymbol{x}'\in\mathbf{T}.$$

Thus, for any given $\boldsymbol{z} \in \mathbf{T}$ the function $G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{x}'+\boldsymbol{z})$ is the solution to the problem

$$-\Big(A^{\varkappa}_{ij}(\boldsymbol{x}/\varepsilon+\boldsymbol{\zeta}+\boldsymbol{z}/\varepsilon)u_{,x_i}(\boldsymbol{x},\boldsymbol{x}')\Big)_{,x_j} = \delta(\boldsymbol{x}-\boldsymbol{x}') - \frac{1}{|\mathbf{T}|}, \quad \int\limits_{\mathbf{T}} u(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{x} = 0. \quad (4.5)$$

Obviously, the problem (4.5) has the function $G^{\zeta+\frac{z}{\varepsilon},\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')$ as its solution, and thus due to uniqueness

$$G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{x}'+\boldsymbol{z}) = G^{\zeta+\frac{z}{\varepsilon},\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}'). \quad (4.6)$$

Integrating the last identity with respect to $\boldsymbol{\zeta}$, we get

$$\int\limits_{Q} G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{x}'+\boldsymbol{z})\mathrm{d}\boldsymbol{\zeta} = \int\limits_{Q} G^{\zeta+\frac{z}{\varepsilon},\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{\zeta}$$

$$= \int\limits_{Q-\frac{z}{\varepsilon}} G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{\zeta} = \int\limits_{Q} G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\mathrm{d}\boldsymbol{\zeta}. \quad (4.7)$$

The last equality in (4.7) is due to Q-periodicity of $G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')$ with respect to $\boldsymbol{\zeta}$.

Hence, we conclude that the average $\Big\langle G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle_{\zeta}$ depends only on the difference $\boldsymbol{x}-\boldsymbol{x}'$. We denote

$$E^{\varepsilon,\varkappa}(\boldsymbol{x}-\boldsymbol{x}') := -\frac{1}{2}\Big\langle G^{\zeta,\varepsilon}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle_{\zeta}.$$

Using identity (4.6) we also get

$$A^{\zeta,\varkappa}_{ij}\Big((\boldsymbol{x}+\boldsymbol{z})/\varepsilon\Big)G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{x}'+\boldsymbol{z}) = A^{\zeta+\frac{z}{\varepsilon},\varkappa}_{ij}(\boldsymbol{x}/\varepsilon)G^{\zeta+\frac{z}{\varepsilon},\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}'),$$

which upon integration with respect to the translation parameter $\boldsymbol{\zeta}$ yields independence of the average $\Big\langle A^{\zeta,\varkappa}_{ij}\Big((\boldsymbol{x}+\boldsymbol{z})/\varepsilon\Big)G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{x}'+\boldsymbol{z})\Big\rangle_{\zeta}$ of $\boldsymbol{z}$. Hence, the average $\Big\langle A^{\zeta,\varkappa}_{ij}(\boldsymbol{x}/\varepsilon)G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle_{\zeta}$ depends only on the difference $\boldsymbol{x}-\boldsymbol{x}'$. We denote

$$\Sigma^{\varepsilon,\varkappa}_{ij}(\boldsymbol{x}-\boldsymbol{x}') := -\Big\langle A^{\zeta,\varkappa}_{ij}(\boldsymbol{x}/\varepsilon)G^{\zeta,\varepsilon,\varkappa}(\boldsymbol{x},\boldsymbol{x}')\Big\rangle_{\zeta}.$$

Thus, the equalities (4.3) and (4.4) can be rewritten as $\bar{e}^{\varepsilon,\varkappa}_j(\boldsymbol{x}) = E^{\varepsilon,\varkappa} * f_{,j}$ and $\bar{\sigma}^{\varepsilon,\varkappa}_i(\boldsymbol{x}) = \Sigma^{\varepsilon,\varkappa}_{ij} * f_{,j}$ respectively, and the infinite-order stress-strain relation can be written in the following formal way

$$\bar{\sigma}^{\varepsilon}_i(\boldsymbol{x}) = \tilde{A}^{\varepsilon,\varkappa}_{ij} * \bar{e}^{\varepsilon}_j(\boldsymbol{x}), \quad (4.8)$$

where $\tilde{A}^{\varepsilon,\varkappa}_{ij} * = \Sigma^{\varepsilon,\varkappa}_{ij} * (E^{\varepsilon,\varkappa}*)^{-1}$. (Resorting *e.g.* to the Fourier transform we can see that the convolution operator $E^{\varepsilon,\varkappa}*$ is invertible and its inverse is a convolution, and therefore the total operator $\Sigma^{\varepsilon,\varkappa}_{ij} * (E^{\varepsilon,\varkappa}*)^{-1}$ is a convolution operator.) The equation (4.8) illustrates the fact that the nature of the infinite-order homogenised stress-strain relation is non-local, although the operators $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$ are far from being explicit. The main point to make in the following sections is that when ε (and possibly $\varkappa$) become small, the non-local operator $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$ has a more explicit "asymptotics" when $\varepsilon \to 0$, which either "localizes" via a "gradient-type" approximation or remains non-local.

4.2 Relation between non-locality for fixed ε and higher-order homogenised equations

As we showed in the previous section (*cf.* (4.8)), for fixed ε the equality (4.8) holds. Now, let the parameter ε in the formula (4.8) tend to zero. In Chapter 1 (see (1.81)) we derived the infinite-order stress-strain relation for the medium under consideration as follows

$$\bar{\sigma}_i^{\varepsilon,\varkappa}(\boldsymbol{x}) \sim 2h_{ij}^{\varkappa}\bar{e}_j^{\varepsilon,\varkappa}(\boldsymbol{x}) + 2\sum_{l=1}^{\infty}\varepsilon^l\sum_{|k|=l}h_{ikj}^{\varkappa}D^k\bar{e}_j^{\varepsilon,\varkappa}(\boldsymbol{x}). \tag{4.9}$$

(Here the superscript $\varkappa$ in $h_{ij}^{\varkappa}$ and $h_{ikj}^{\varkappa}$ denotes, as before, dependence of these constants on the parameter $\varkappa$.) The asymptotics (4.9) together with the non-local stress-strain relation (4.8) mean that formally, when $\varepsilon \to 0$ the following asymptotic expansion for the operator $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$ holds

$$\tilde{A}_{ij}^{\varepsilon,\varkappa}* \sim 2h_{ij}^{\varkappa} + 2\sum_{l=1}^{\infty}\varepsilon^l\sum_{|k|=l}h_{ikj}^{\varkappa}D^k. \tag{4.10}$$

This means that in some sense, the non-local operator $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$ asymptotically "localizes" when $\varepsilon \to 0$ into an asymptotic expansion in the form of a higher-gradient series. The analysis of Chapter 1 provides in effect a rigorous *a posteriori* justification of such asymptotics.

Note that the right-hand side of (4.10) is of the form suggested by the so-called "gradient approximation" of the non-local operator $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$. The classical homogenisation theorem in effect states that $\tilde{A}_{ij}^{\varepsilon,\varkappa}(\boldsymbol{x}-\boldsymbol{x}') \longrightarrow 2h_{ij}^{\varkappa}\delta(\boldsymbol{x}-\boldsymbol{x}')$ when $\varepsilon \to 0$. In other words, for small ε the kernel of the non-local operator as a function of $\boldsymbol{x}'$ becomes negligible outside a small neighbourhood of the point $\boldsymbol{x}' = \boldsymbol{x}$. Assuming that $\bar{e}^{\varepsilon,\varkappa}(\boldsymbol{x}')$ varies smoothly in this neighbourhood, it can be well approximated there by its Taylor series

$$\bar{e}^{\varepsilon,\varkappa}(\boldsymbol{x}') \sim \bar{e}^{\varepsilon,\varkappa}(\boldsymbol{x}) + \sum_{l=1}^{\infty}\sum_{|k|=l}\frac{1}{k!}(\boldsymbol{x}-\boldsymbol{x}')^kD^k\bar{e}^{\varepsilon,\varkappa}(\boldsymbol{x}),\text{[1]}$$

which is the asymptotic expansion of $\bar{e}^{\varepsilon,\varkappa}$ in this neighbourhood. Substituting this

[1]Here we use the notation $(\boldsymbol{x}-\boldsymbol{x})^k := (x_{k_1}-x'_{k_1})...(x_{k_l}-x'_{k_l})$.

formally into the convolution operator, we obtain its gradient approximation:

$$\left(\tilde{A}_{ij}^{\varepsilon,\varkappa} * \bar{e}_j^{\varepsilon,\varkappa}\right)(\boldsymbol{x}) \sim T_{ij}^{\varepsilon,\varkappa}\bar{e}_j^{\varepsilon,\varkappa} + \sum_{l=1}^{\infty}\sum_{|k|=l} T_{ikj}^{\varepsilon,\varkappa} D^k \bar{e}_j^{\varepsilon,\varkappa}(\boldsymbol{x}), \tag{4.11}$$

where the "moments" $T_{ikj}^{\varepsilon,\varkappa}$, $|k| \geq 0$, are as follows

$$T_{ikj}^{\varepsilon,\varkappa} := \frac{1}{k!}\int_{\mathbf{T}} \boldsymbol{z}^k \tilde{A}_{ij}^{\varepsilon,\varkappa}(\boldsymbol{z})\mathrm{d}\boldsymbol{z}.$$

Note that the gradient approximation (4.11) is of the same form as (4.10). It is likely therefore that the right-hand side of (4.10) is related to the gradient approximation of the non-local operator $\tilde{A}_{ij}^{\varepsilon,\varkappa}*$ for small ε.

4.3 Non-locality in the limit when $\varepsilon \to 0$ is an essential feature of double-porosity models

To provide an interpretation of the results of Chapter 3 from the point of view of the non-local stress-strain relations derived in Section 4.1, and to see how those results are related to the higher-gradient approach of Chapter 1, we consider the following two-dimensional linear problem with rapidly oscillating coefficients (*cf.* (3.2), (4.1))

$$-\left(A_{ij}^{\varkappa}(\boldsymbol{x}/\varepsilon)u_{,j}^{\varepsilon,\varkappa}\right)_{,i} = f(\boldsymbol{x}), \quad \int_{\mathbf{T}} u^{\varepsilon,\varkappa}\mathrm{d}\boldsymbol{x} = 0, \tag{4.12}$$

where the matrix-valued function $\left(A_{ij}^{\varkappa}(\boldsymbol{y})\right)$ is defined by the following formula (*cf.* (3.1))

$$\left(A_{ij}^{\varkappa}(\boldsymbol{y})\right) = \begin{cases} \begin{pmatrix} \varkappa & 0 \\ 0 & \varkappa \end{pmatrix} & \text{if } \boldsymbol{y} \in F_0, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: I & \text{if } \boldsymbol{y} \in F_1. \end{cases} \tag{4.13}$$

Here $\varkappa > 0$ and F_0, F_1 are Q-periodic sets such that $F_0 \cap Q$ is a smooth subdomain of the unit square Q and disjoint sets $F_0 \cap Q$ and $F_1 \cap Q$ form a partition of the square Q.

In the case when $\varkappa = \varepsilon^2$, the problem (3.2) represents a classical double-porosity (see *e.g.* Allaire [4], Sandrakov [39], [40]). It is well known (see the above cited papers) that $u^{\varepsilon,\varepsilon^2}(\boldsymbol{x})$ has a "two-scale" limit when $\varepsilon \to 0$: $u^{\varepsilon,\varepsilon^2}(\boldsymbol{x}) \overset{2}{\rightharpoonup} u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x},\boldsymbol{y})$, where the functions $u^{(1)}$ and w satisfy the following system of elliptic equations

$$\begin{cases} -\mathrm{div}\left(A_{2D}^{hom}\nabla u^{(1)}\right) = f, & \boldsymbol{x} \in \mathbf{T}, \\ -\Delta_{\bar{y}} w = f, & \boldsymbol{y} \in F_0 \cap Q, \end{cases} \tag{4.14}$$

together with the conditions

$$w(\boldsymbol{x},\boldsymbol{y})|_{\tilde{y}\in\partial F_0\cap Q}=0,\qquad \int\limits_{\mathbf{T}}\Big(u^{(1)}(\boldsymbol{x})+\langle w\rangle(\boldsymbol{x})\Big)\mathrm{d}\boldsymbol{x}=0.$$

In contrast to our non-local two-scale homogenised system (3.25) [Remind that the equation for $u^{(1)}$ obtained from (3.25) upon elimination of w is non-local — see (3.57).], the system (4.14) is local. However, it exhibits non-locality in the sense of Section 3.3.3, *i.e.* the "limiting" stress-strain relation corresponding to the system (4.14) is non-local. We use this fact to illustrate how non-locality can "survive" homogenisation in double-porosity models, as contrasts to the uniformly elliptic problems, whose homogenised limit is a (local) differential equation. To this end, we first fix ε and $\varkappa$ in the problem (4.12). Then, as was shown in the Section 4.1, the corresponding stress-strain relation is

$$\bar{\boldsymbol{\sigma}}^{\varepsilon,\varkappa}=\tilde{A}^{\varepsilon,\varkappa}*\bar{\boldsymbol{e}}^{\varepsilon,\varkappa}.$$

On the other hand, we know from the previous asymptotic analysis (Section 3.3.3) that for any $\boldsymbol{\zeta}\in Q$ the following convergences hold $\boldsymbol{e}^{\zeta,\varepsilon,\varepsilon^2}\to\boldsymbol{e}=\nabla u^{(1)}+\nabla\langle w\rangle$ in $\left[H^{-1}_{per}(\mathbf{T})\right]^3$ weakly and $\boldsymbol{\sigma}^{\zeta,\varepsilon,\varepsilon^2}\to\boldsymbol{\sigma}=A^{hom}_{2D}\nabla u^{(1)}$ in $\left[L^2(\mathbf{T})\right]^3$ weakly. Hence, the same convergences hold for the ensemble averages $\bar{\boldsymbol{\sigma}}^{\varepsilon,\varepsilon^2}$ and $\bar{\boldsymbol{e}}^{\varepsilon,\varepsilon^2}$. We also know (see formula (3.117)) that the vectors $\boldsymbol{e}$ and $\boldsymbol{\sigma}$ are linked by a non-local stress-strain relation as follows

$$\boldsymbol{\sigma}=A^{hom}_{2D}\boldsymbol{e}+h_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left[\langle g\rangle_{\tilde{y},\tilde{y}'}\overset{x_3}{*}\Phi*\begin{pmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{pmatrix}\boldsymbol{e}\right]=:A^{hom}_{2D}\boldsymbol{e}+\mathcal{A}\boldsymbol{e},\tag{4.15}$$

where (*cf.* (3.59))

$$h_{ij}=\int\limits_{\tilde{F}_1\cap Q_2}\Big(\delta_{ij}+(N_i)_{,j}(\tilde{\boldsymbol{y}})\Big)\mathrm{d}\tilde{\boldsymbol{y}},\quad i,j=1,2,\tag{4.16}$$

and $g=g(\boldsymbol{y},\boldsymbol{y}')$ is the Green's function of the Laplacian in the domain $F_0\cap Q$. Thus, in some sense,

$$\tilde{A}^{\varepsilon,\varepsilon^2}*\to A^{hom}_{2D}+\mathcal{A}\ \text{ as }\ \varepsilon\to 0.$$

Notice that for fixed $\varkappa$ we have $\tilde{A}^{\varepsilon,\varkappa}*\to A^{hom,\varkappa}_{2D}:=h^{\varkappa}_{ij}$ as $\varepsilon\to 0$, with higher-order correctors according to the previous section. Thus for finite $\varkappa$ the limit of the matrix operator $\tilde{A}^{\varepsilon,\varkappa}*$ when $\varepsilon\to 0$, is local. From this perspective, the double porosity effect ($\varkappa(\varepsilon)=\varepsilon^2$) means that the non-locality in $\tilde{A}^{\varepsilon,\varkappa(\varepsilon)}*$ remains to the "very limit" as $\varepsilon\to 0$. In the next section we show how this can also be seen from passing to the limit as $\varepsilon\to 0$ in the gradient-type approximation (4.10).

4.4 A formal derivation of two-scale limits in the double-porosity model from the higher-gradient approach

In this section we show how the system (4.14) (and thus the non-local stress-strain relation (4.15)) can be formally derived from the full asymptotic expansion of the solution to a rapidly oscillating problem, which was presented in Chapter 1. Thus the gradient approximation (4.10) considered as a function of the parameter $\varkappa$ carries important information about a range of non-uniformly elliptic problems.

To see this, consider the problem (4.12). Under the assumptions on the matrix function $\left(A_{ij}^{\varkappa}(\boldsymbol{y})\right)$, it is piecewise smooth in Q and therefore we can apply an asymptotic procedure presented in the book by Bakhvalov and Panasenko [11] (see also Smyshlyaev & Cherednichenko [42]). Namely, for the solution to the problem (4.12) the following two-scale asymptotic expansion holds

$$u^{\varepsilon,\varkappa}(\boldsymbol{x}) \sim v^{\varkappa}(\boldsymbol{x},\varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{|k|=l} N_k^{\varkappa}(\boldsymbol{x}/\varepsilon) D^k v^{\varkappa}(\boldsymbol{x},\varepsilon), \tag{4.17}$$

where

$$v^{\varkappa}(\boldsymbol{x},\varepsilon) \sim \sum_{s=0}^{\infty} \varepsilon^s v_s^{\varkappa}(\boldsymbol{x}). \tag{4.18}$$

Here, the microstructural functions $N_k^{\varkappa}(\boldsymbol{y})$ are solutions to the associated unit cell problems. The superscript $\varkappa$ stands for the fact that the corresponding objects depend on the parameter $\varkappa$.

The function $v^{\varkappa}(\boldsymbol{x},\varepsilon)$ solves the averaged equation of infinite order, which is as follows (*cf.* (1.100))

$$-h_{ij}^{\varkappa} v_{,ij}^{\varkappa} - \sum_{l=3}^{\infty} \varepsilon^{l-2} \sum_{|k|=l} h_k^{\varkappa} D^k v^{\varkappa} \sim f, \tag{4.19}$$

where the constants $h_k^{\varkappa}$ are expressed in terms of the functions $N_k^{\varkappa}(\boldsymbol{y})$.

After performing some formal manipulations on the expression in the left-hand side of (4.19) we notice that for $\varkappa \sim \varepsilon^2$ the asymptotic series (4.17) and (4.19) "break up": in each of them all the terms become of "equal strength", *i.e.* the series cease to be asymptotical. Namely, retaining the leading contributions in each term of (4.19) when $\varkappa \sim \varepsilon^2$ we arrive at the following formal equation

$$-h_{ij} v_{,ij}^{\varkappa}(\boldsymbol{x}) - A\varepsilon^2 \varkappa^{-1} h_{ij} h_{rs} v_{,ijrs}^{\varkappa}(\boldsymbol{x}) - A^2 \varepsilon^4 \varkappa^{-2} h_{ij} h_{rs} h_{pt} v_{,ijrspt}^{\varkappa}(\boldsymbol{x}) + \ldots$$

$$\equiv -\sum_{m=0}^{\infty} A^m \varepsilon^{2m} \varkappa^{-m} \sum_{|k|=2m+2} h_{k_1k_2} h_{k_3k_4} \ldots h_{k_{2m+1}k_{2m+2}} v_k^{\varkappa}(\boldsymbol{x}) \sim f, \tag{4.20}$$

where the coefficients h_{ij} are given by the formula (4.16). In the formula (4.20) the following notation is used

$$A := \int\limits_{F_0 \cap Q} \int\limits_{F_0 \cap Q} g(\boldsymbol{y}, \boldsymbol{y}')\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}',$$

where $g(\boldsymbol{y}, \boldsymbol{y}')$ is the Green's function of the Laplacian in the domain $F_0 \cap Q$. Clearly, every term in the left-hand side of (4.20) is of order one. To simplify this expression we denote its truncation at $m = K$ by $-M_{ij}^K(\varepsilon, \varkappa)v^{\varkappa}_{,ij}$, so that the differential operators $M_{ij}^K(\varepsilon, \varkappa)$ are as follows

$$M_{ij}^K(\varepsilon, \varkappa) = h_{ij} \sum_{m=0}^{K} A^m \varepsilon^{2m} \varkappa^{-m} \sum_{|k|=2m} h_{k_1k_2} h_{k_3k_4} ... h_{k_{2m-1}k_{2m}} D^k.$$

It is not difficult to see that the following identity holds

$$M_{ij}^K(\varepsilon, \varkappa) = h_{ij} + A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big(M_{ij}^K(\varepsilon, \varkappa) - h_{ij}\Big(A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big)^K\Big). \tag{4.21}$$

Formally inverting (4.21) we obtain the following formula for the operator $M_{ij}^K(\varepsilon, \varkappa)$:

$$M_{ij}^K(\varepsilon, \varkappa) = \Big(I - A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big)^{-1}\Big(I - \Big(A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big)^{K+1}\Big)h_{ij}. \tag{4.22}$$

Setting $\varkappa = \alpha^{-1}\varepsilon^2$ in the right-hand side of (4.22) we arrive at

$$M_{ij}^K(\varepsilon, \varkappa) = -\Big(I - \alpha A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big)^{-1}\Big(I - \Big(\alpha A h_{pr}D^{pr}\Big)^{K+1}\Big)h_{ij}.$$

Finally passing formally to the limit when $K \to \infty$ in the last expression we get (for sufficiently small α):

$$M_{ij}^K(\varepsilon, \varkappa) = -\Big(I - \alpha A\varepsilon^2\varkappa^{-1}h_{pr}D^{pr}\Big)^{-1} h_{ij},$$

and thus, using (4.20) we obtain the following equation for the leading term $v(\boldsymbol{x})$ of the function $v^{\alpha^{-1}\varepsilon^2}$:

$$-h_{ij}v_{,ij} = f - \alpha A h_{ij} f_{,ij}. \tag{4.23}$$

In a similar fashion we consider the leading term of the expansion (4.17) for arbitrary $\varkappa$ and ε such that $\varkappa \sim \varepsilon^2$, perform some formal manipulations and then substitute $\varkappa = \varepsilon^2$. We get the following expression for the leading term of the asymptotics (4.17)

$$u^{\varepsilon,\varepsilon^2}(\boldsymbol{x}) \sim u^{(1)}(\boldsymbol{x}) + w(\boldsymbol{x}, \boldsymbol{x}/\varepsilon), \tag{4.24}$$

where

$$u^{(1)} = v + \alpha A h_{ij} u^{(1)}_{,ij} \tag{4.25}$$

and

$$w(\boldsymbol{x},\boldsymbol{y}) = -\alpha h_{ij}\Big(u^{(1)}(\boldsymbol{x})\Big)_{,ij} \int_{F_0\cap Q} g(\boldsymbol{y},\boldsymbol{y}')\mathrm{d}\boldsymbol{y}'. \tag{4.26}$$

To conclude the derivation note that for $\alpha = 1$, the formulas (4.23), (4.25) and (4.26) imply the system (4.14), while the corresponding stress-strain relation is non-local and is given by the formula (4.15). Thus, the double-porosity equations are formally recovered from the formal asymptotic expansions (4.17)–(4.18) and (4.19). Since the double porosity equations in fact hold for $\alpha = 1$, it may happen that the above series actually converge for finite α.

Discussion

Non-local constitutive relations akin to the those derived in Chapter 3, Section 3.3.3 have a general nature and are peculiar to the overall behaviour of periodic heterogeneous media. The formula (4.8) suggests a general form of such a constitutive relation, which is given by certain convolution operators $\tilde{A}^{\varepsilon,\varkappa}_{ij}*$ for given cell size ε and contrast $\varkappa$.

One of the key features of the double porosity models (for which $\varkappa(\varepsilon) = \varepsilon^2$) introduced in Chapter 2 is that constitutive laws for such models are non-local not only for fixed $\varepsilon > 0$ but also in the homogenised limit (when $\varepsilon \to 0$).

Importantly, the higher-order homogenised constitutive relations of the uniformly elliptic case studied thoroughly in Chapter 1 and Chapter 2 contain a certain amount of this "non-local information" about the double porosity case. This can be seen by considering the "gradient-type approximation" of the operators $\tilde{A}^{\varepsilon,\varkappa}_{ij}*$, taking $\varkappa = \varepsilon^2$ in it, and passing to the limit with respect to the small parameter ε.

On the other hand, let $\alpha := \varepsilon^2\varkappa$, so the double porosity case corresponds to finite α and $\varepsilon \to 0$. (In what we considered before $\alpha = 1$, but the analysis extends for any other "finite" α.) Then the non-local double-porosity homogenised operator $\mathcal{A}^\alpha$, $\alpha = \varepsilon^2\varkappa^{-1}$, which equals $\mathcal{A}$ for $\alpha = 1$ (see (4.15)), "localizes" for small α into a gradient-type approximation consistent with retaining in the higher-order homogenised relations of Chapter 1 the main order terms in $\varepsilon^2\varkappa^{-1}$.

This all provides in our opinion some unified view on the higher-gradient and non-local effects in the overall behaviour of rapidly oscillating periodic media.

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